Supplementary Appendix for Learning and Expectations in Dynamic Spatial Economies

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This appendix presents non-essential supplementary materials.

Contents

Appendix C Additional Derivations for the One-Sector Model

In this section, we derive the first and second-order approximations of equations (2)-(7) for the one-sector model.

C.1 First-Order Derivatives

First-order approximation to value function

Let $v_{nt+1} = \left(U(w_{nt+1}, P_{nt+1}) + \nu \ln \left[\sum_{i=1}^{N} \exp(\beta v_{it+2} - m_{nit+1})^{1/\nu} \right] \right) \equiv F_n(w_{nt+1}, P_{nt+1}, v_{t+2})$. Then, a first-order approximation of $\mathbb{E}_t \hat{v}_{nt+1}$ is:

$$\begin{aligned}
\mathbb{E}_t \hat{v}_{nt+1} \quad &\approx \quad \frac{\partial F_n}{\partial \ln(\bar{w}_{nt+1})} \mathbb{E}_t \hat{w}_{nt+1} + \frac{\partial F_n}{\partial \ln(\bar{P}_{nt+1})} \mathbb{E}_t \hat{P}_{nt+1} + \sum_{i=1}^{N} \frac{\partial F_n}{\partial \bar{v}_{it+2}} \mathbb{E}_t \hat{v}_{it+2} \\
&= \quad \frac{\partial U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{w}_{nt+1})} \mathbb{E}_t \hat{w}_{nt+1} + \frac{\partial U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{P}_{nt+1})} \mathbb{E}_t \hat{P}_{nt+1} + \beta \sum_{i=1}^{N} \bar{\mu}_{nit+1} \mathbb{E}_t \hat{v}_{it+2}. \qquad \text{(C.1)}
\end{aligned}$$

First-order approximation to mobility flows
Let $\ln \mu_{nit} = \ln \left(\frac{\exp(\beta v_{it+1} - m_{nit})^{1/\nu}}{\sum_{h=1}^{N} \exp(\beta v_{ht+1} - m_{nht})^{1/\nu}} \right) \equiv G_{ni}(v_{t+1})$. Then, a first-order approximation of $\mathbb{E}_t \hat{\mu}_{nit}$ is:

$$\begin{aligned} \mathbb{E}_t \hat{\mu}_{nit} &\approx \sum_{m=1}^{N} \frac{\partial G_{ni}}{\partial \bar{v}_{mt+1}} \mathbb{E}_t \hat{v}_{mt+1} \\ &= \frac{\beta}{\nu} \sum_{m=1}^{N} \Big(\mathbb{1}(m=i) - \bar{\mu}_{nmt} \Big) \mathbb{E}_t \hat{v}_{mt+1}. \end{aligned} \tag{C.2}$$

First-order approximation to law of motion of labor
Let $\ln l_{nt+1} = \ln \sum_{i=1}^{N} \mu_{int} l_{it} \equiv J(\mu_{nt+1}, l_t)$ where $\mu_{nt} = [\mu_{1nt}, ..., \mu_{Nnt}]'$. Then, a first-order approximation of $\mathbb{E}_t \hat{l}_{nt+1}$ is:

$$\begin{aligned} \mathbb{E}_t \hat{l}_{nt+1} &\approx \sum_{i=1}^{N} \Big(\frac{\partial J}{\partial \ln(\bar{\mu}_{int})} \mathbb{E}_t \hat{\mu}_{int} + \frac{\partial J}{\partial \ln(\bar{l}_{it})} \mathbb{E}_t \hat{l}_{it} \Big) \\ &= \sum_{i=1}^{N} \frac{\bar{\mu}_{int} \bar{l}_{it}}{\bar{l}_{nt+1}} \Big(\mathbb{E}_t \hat{\mu}_{int} + \mathbb{E}_t \hat{l}_{it} \Big). \end{aligned} \tag{C.3}$$

First-order approximation to trade shares
Let $\ln \lambda_{nit} = \ln \left[z_{it} \left(\frac{w_{it} \kappa_{nit}}{P_{nt}} \right)^{-\theta} \right] \equiv K(w_{it}, \kappa_{nit}, P_{nt}, z_{it})$. Then, a first-order approximation of $\mathbb{E}_t \hat{\lambda}_{nit}$ is:

$$\begin{aligned} \mathbb{E}_t \hat{\lambda}_{nit} &= \Big(\frac{\partial K}{\partial \ln(\bar{w}_{it})} \mathbb{E}_t \hat{w}_{it} + \frac{\partial K}{\partial \ln(\bar{\kappa}_{nit})} \mathbb{E}_t \hat{\kappa}_{int} + \frac{\partial K}{\partial \ln(\bar{P}_{nt})} \mathbb{E}_t \hat{P}_{nt} + \frac{\partial K}{\partial \ln(\bar{z}_{it})} \hat{z}_{it} \Big) \\ &= -\theta (\mathbb{E}_t \hat{w}_{it} + \mathbb{E}_t \hat{\kappa}_{nit} - \mathbb{E}_t \hat{P}_{nt}) + \hat{z}_{it}. \end{aligned} \tag{C.4}$$

First-order approximation to price index
Let $\ln P_{nt} = \ln \Big(\sum_{i=1} z_{it} (w_{it} \kappa_{nit})^{-\theta} \Big)^{-\frac{1}{\theta}} \equiv L_n(w_t, \kappa_{nt}, z_t)$ where $\kappa_{nt} = [\kappa_{n1t}, ..., \kappa_{nNt}]'$. Then, a first-order approximation of $\mathbb{E}_t \hat{P}_{nt}$ is:

$$\begin{aligned} \mathbb{E}_t \hat{P}_{nt} &\approx \sum_{i=1}^{N} \Big(\frac{\partial L_n}{\partial \ln(\bar{w}_{it})} \mathbb{E}_t \hat{w}_{it} + \frac{\partial L_n}{\partial \ln(\bar{\kappa}_{nit})} \mathbb{E}_t \hat{\kappa}_{nit} + \frac{\partial L_n}{\partial \ln(\bar{z}_{it})} \hat{z}_{it} \Big) \\ &= \sum_{i=1}^{N} \bar{\lambda}_{nit} \Big(\mathbb{E}_t \hat{w}_{it} + \mathbb{E}_t \hat{\kappa}_{nit} - \frac{1}{\theta} \hat{z}_{it} \Big). \end{aligned} \tag{C.5}$$

First-order approximation to labor market clearing
Let $\ln \Big(w_{nt} l_{nt} \Big) = \ln \Big(\sum_{i=1}^{N} \lambda_{int} w_{it} l_{it} \Big) \equiv M_n(\lambda_{nt}, w_t, l_t)$ where $\lambda_{nt} = [\lambda_{1nt}, ..., \lambda_{Nnt}]'$. Then, a first-order approximation of $\mathbb{E}_t \hat{w}_{it}$ is:

$$\begin{aligned} \mathbb{E}_t \hat{w}_{nt} + \mathbb{E}_t \hat{l}_{nt} &\approx \sum_{i=1}^{N} \Big(\frac{\partial M_n}{\partial \ln(\bar{\lambda}_{nit})} \hat{\lambda}_{int} + \frac{\partial M_n}{\partial \ln(\bar{w}_{it})} \mathbb{E}_t \hat{w}_{it} + \frac{\partial M_n}{\partial \ln(\bar{l}_{it})} \mathbb{E}_t \hat{l}_{it} \Big) \\ &= \sum_{i=1}^{N} \frac{\bar{\lambda}_{int} \bar{w}_{it} \bar{l}_{it}}{\bar{w}_{nt} \bar{l}_{nt}} \Big(\mathbb{E}_t \hat{\lambda}_{int} + \mathbb{E}_t \hat{w}_{it} + \mathbb{E}_t \hat{l}_{it} \Big). \end{aligned} \tag{C.6}$$

C.2 Second-Order Derivatives

Second-order approximation to value function

Let $v_{nt+1} = \left(U(w_{nt+1}, P_{nt+1}) + \nu \ln \left[\sum_{i=1}^{N} \exp(\beta v_{it+2} - m_{nit+1})^{1/\nu} \right] \right) \equiv F_n(w_{nt+1}, P_{nt+1}, v_{t+2})$. Then, a second-order approximation of $\mathbb{E}_t \hat{v}_{nt+1}$ is given by

$$\begin{aligned}
\mathbb{E}_t \hat{v}_{nt+1} \approx & \frac{\partial F_n}{\partial \ln(\bar{w}_{nt+1})} \mathbb{E}_t \hat{w}_{nt+1} + \frac{\partial F_n}{\partial \ln(\bar{P}_{nt+1})} \mathbb{E}_t \hat{P}_{nt+1} + \frac{\partial^2 F_n}{\partial \ln(\bar{w}_{nt+1}) \partial \ln(\bar{P}_{nt+1})} \mathbb{E}_t \hat{w}_{nt+1} \hat{P}_{nt+1} \\
& + \frac{1}{2} \frac{\partial^2 F_n}{\partial \ln(\bar{w}_{nt+1})^2} \mathbb{E}_t \hat{w}_{nt+1}^2 + \frac{1}{2} \frac{\partial^2 F_n}{\partial \ln(\bar{P}_{nt+1})^2} \mathbb{E}_t \hat{P}_{nt+1}^2 \\
& + \sum_{m=1}^{N} \frac{\partial F_n}{\partial \bar{v}_{mt+2}} \mathbb{E}_t \hat{v}_{mt+2} + \frac{1}{2} \sum_{o,h}^{N} \frac{\partial^2 F_n}{\partial \bar{v}_{ot+2} \partial \bar{v}_{ht+2}} \mathbb{E}_t \hat{v}_{ot+2} \hat{v}_{ht+2}. \qquad \text{(C.7)}
\end{aligned}$$

where

$$\begin{aligned}
& \frac{\partial F_n}{\partial \ln(\bar{w}_{nt+1})} = \frac{\partial U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{w}_{nt+1})}, \quad \frac{\partial F_n}{\partial \ln(\bar{P}_{nt+1})} = \frac{\partial U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{P}_{nt+1})}, \\
& \frac{\partial^2 F_n}{\partial \ln(\bar{w}_{nt+1})^2} = \frac{\partial^2 U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{w}_{nt+1})^2}, \quad \frac{\partial^2 F_n}{\partial \ln(\bar{P}_{nt+1})^2} = \frac{\partial^2 U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{P}_{nt+1})^2}, \\
& \frac{\partial^2 F_n}{\partial \ln(\bar{w}_{nt+1}) \partial \ln(\bar{P}_{nt+1})} = \frac{\partial^2 U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{w}_{nt+1}) \partial \ln(\bar{P}_{nt+1})}, \frac{\partial F_n}{\partial \bar{v}_{mt+2}} = \beta \bar{\mu}_{nmt+1}, \text{ and} \\
& \frac{\partial^2 F_i}{\partial \bar{v}_{ot+2} \partial \bar{v}_{ht+2}} = \beta \frac{\partial \bar{\mu}_{not+1}}{\partial \bar{v}_{ht+2}} = \frac{\beta^2}{\nu} \bar{\mu}_{not+1} \Big(\mathbb{1}(o = h) - \bar{\mu}_{not+1} \Big).
\end{aligned}$$

Second-order approximation to mobility flows

Let $\ln \mu_{nit} = \ln \frac{\exp(\beta v_{it+1} - m_{nit})^{1/\nu}}{\sum_{h=1}^{N} \exp(\beta v_{ht+1} - m_{nht})^{1/\nu}} \equiv G_{ni}(v_{t+1})$. A second-order approximation of $\mathbb{E}_t \hat{\mu}_{nit}$ is:

$$\mathbb{E}_t \hat{\mu}_{nit} \approx \sum_{m=1}^{N} \frac{\partial G_{ni}}{\partial \bar{v}_{mt+1}} \mathbb{E}_t \hat{v}_{mt+1} + \frac{1}{2} \sum_{m,o}^{N} \frac{\partial^2 G_{ni}}{\partial \bar{v}_{mt+1} \partial \bar{v}_{ot+1}} \mathbb{E}_t \hat{v}_{mt+1} \hat{v}_{ot+1}, \qquad \text{(C.8)}$$

where

$$\begin{aligned}
\frac{\partial G_{ni}}{\partial \bar{v}_{mt+1}} &= \frac{\beta}{\nu} \Big(\mathbb{1}(m = i) - \bar{\mu}_{nmt} \Big) \\
\frac{\partial^2 G_{ni}}{\partial \bar{v}_{mt+1} \partial \bar{v}_{ot+1}} &= \frac{\beta}{\nu} \Big(\mathbb{1}(m = i) \frac{\partial \bar{\mu}_{nit}}{\partial \bar{v}_{ot+1}} - \bar{\mu}_{nmt} \frac{\partial \bar{\mu}_{nit}}{\partial \bar{v}_{ot+1}} - \bar{\mu}_{nit} \frac{\partial \bar{\mu}_{nmt}}{\partial \bar{v}_{ot+1}} \Big) \\
&= \left(\frac{\beta}{\nu} \right)^2 \Big[\mathbb{1}(m = i) \bar{\mu}_{nit} (\mathbb{1}(o = i) - \bar{\mu}_{not}) - \bar{\mu}_{nmt} \bar{\mu}_{nit} [\mathbb{1}(o = i) - \bar{\mu}_{not} + \mathbb{1}(o = m) - \bar{\mu}_{not}] \Big].
\end{aligned}$$

Second-order approximation to law of motion of labor

Let $\ln l_{nt+1} = \ln \sum_{i=1}^{N} \mu_{int}(v_{nt+1}) l_{it} \equiv J_n(v_{t+1}, l_t)$. For notational ease, let $\bar{\psi}_{int+1} = \frac{\bar{\mu}_{int} \bar{l}_{it}}{\bar{l}_{nt+1}}$. Then, a second-order approximation of $\mathbb{E}_t \hat{l}_{nt+1}$ is:

$$\begin{aligned}
\mathbb{E}_t \hat{l}_{nt+1} \approx & \sum_{m=1}^{N} \frac{\partial J_n}{\partial \bar{v}_{mt+1}} \hat{v}_{mt+1} + \sum_{k=1}^{N} \frac{\partial J_n}{\partial \ln(\bar{l}_{mt})} \mathbb{E}_t \hat{l}_{mt} + \frac{1}{2} \sum_{m,o}^{N} \frac{\partial^2 J_n}{\partial \bar{v}_{mt+1} \partial \bar{v}_{ot+1}} \mathbb{E}_t \hat{v}_{mt+1} \hat{v}_{ot+1} \\
& + \frac{1}{2} \sum_{m,o}^{N} \frac{\partial^2 J_n}{\partial \ln(\bar{l}_{mt}) \partial \ln(\bar{l}_{ot})} \mathbb{E}_t \hat{l}_{mt} \hat{l}_{ot} + \sum_{m,o}^{N} \frac{\partial^2 J_n}{\partial \bar{v}_{mt+1} \partial \ln(\bar{l}_{ot})} \mathbb{E}_t \hat{v}_{mt+1} \hat{l}_{ot}, \qquad \text{(C.9)}
\end{aligned}$$

where

$$
\begin{aligned}
\frac{\partial J_n}{\partial \bar{v}_{mt+1}} &= \frac{\beta}{\nu}\sum_{i=1}^{N}\bar{\psi}_{int+1}\Big(\mathbb{1}(m=n)-\bar{\mu}_{imt}\Big)\\
\frac{\partial J_n}{\partial \ln(\bar{l}_{mt})} &= \frac{\bar{\mu}_{mnt}\bar{l}_{mt}}{\bar{l}_{nt+1}} \equiv \bar{\psi}_{mnt+1}\\
\frac{\partial^2 J_n}{\partial \ln(\bar{l}_{mt})\partial \ln(\bar{l}_{ot})} &= \mathbb{1}(m=o)\bar{\psi}_{ont+1}-\bar{\psi}_{mnt+1}\bar{\psi}_{ont+1}\\
\frac{\partial^2 J_n}{\partial \bar{v}_{mt+1}\partial \bar{v}_{ot+1}} &= \frac{\partial^2 J_n}{\partial \bar{v}_{ot+1}\partial \bar{v}_{mt+1}} = \frac{\partial \frac{\partial J_n}{\partial \bar{v}_{mt+1}}}{\partial \bar{v}_{ot+1}} = \frac{\partial \frac{\frac{\beta}{\nu}\sum_{i=1}^{N}\bar{l}_{it}\bar{\mu}_{int}\Big(\mathbb{1}(m=n)-\bar{\mu}_{imt}\Big)}{\bar{l}_{nt+1}}}{\partial \bar{v}_{ot+1}}\\
&= \left(\frac{\beta}{\nu}\right)^2\Big[-\sum_{i=1}^{N}\bar{\mu}_{imt}\bar{\psi}_{int+1}[\mathbb{1}(o=n)+\mathbb{1}(o=m)-2\bar{\mu}_{iot}]\\
&\quad+\mathbb{1}(k=n)\sum_{i}\bar{\psi}_{int+1}(\mathbb{1}(o=n)-\bar{\mu}_{iot})\\
&\quad-\Big[\sum_{i=1}^{N}\bar{\psi}_{int+1}\big(\mathbb{1}(m=n)-\bar{\mu}_{imt}\big)\Big]\Big[\sum_{i=1}^{N}\bar{\psi}_{int+1}\big(\mathbb{1}(o=n)-\bar{\mu}_{iot}\big)\Big]\Big]\\
\frac{\partial^2 J_n}{\partial \bar{v}_{mt+1}\partial \ln(\bar{l}_{ot})} &= \frac{\beta}{\nu}\bar{\psi}_{ont+1}\big(\mathbb{1}(m=n)-\bar{\mu}_{omt}\big)-\frac{\beta}{\nu}\bar{\psi}_{ont+1}\Big[\sum_{i=1}^{N}\bar{\psi}_{int+1}\big(\mathbb{1}(m=n)-\bar{\mu}_{imt}\big)\Big].
\end{aligned}
$$

Second-order approximation to trade share

First order approximation for trade share $\boldsymbol{\lambda}$ is exact. Therefore,

$$
\begin{aligned}
\mathbb{E}_t\hat{\lambda}_{nit} &= \Big(\frac{\partial K}{\partial \ln(\bar{w}_{it})}\mathbb{E}_t\hat{w}_{it+1}+\frac{\partial K}{\partial \ln(\bar{\kappa}_{int})}\mathbb{E}_t\hat{\kappa}_{int}+\frac{\partial K}{\partial \ln(\bar{P}_{nt})}\mathbb{E}_t\hat{P}_{nt}+\frac{\partial K}{\partial \ln(\bar{z}_{nt})}\mathbb{E}_t\hat{z}_{nt}\Big)\\
&= -\theta(\mathbb{E}_t\hat{w}_{it}+\mathbb{E}_t\hat{\kappa}_{int}-\mathbb{E}_t\hat{P}_{nt})+\mathbb{E}_t\hat{z}_{it}.
\end{aligned}
\tag{C.10}
$$

Second-order approximation to price index

Let $\ln P_{nt} = \ln\Big(\sum_{i=1}(w_{it}\kappa_{nit})^{-\theta}z_{it}\Big)^{-\frac{1}{\theta}} \equiv L_n(w_t,\kappa_{nt},z_t)$. Then, a second-order approximation of $\mathbb{E}_t\hat{P}_{nt}$ is:

$$
\begin{aligned}
\mathbb{E}_t\hat{P}_{nt} \approx\; & \sum_{i=1}^{N}\frac{\partial L_n}{\partial \ln(\bar{w}_{it})}\mathbb{E}_t\hat{w}_{it}+\sum_{i=1}^{N}\frac{\partial L_n}{\partial \ln(\bar{\kappa}_{nit})}\mathbb{E}_t\hat{\kappa}_{nit}+\sum_{i=1}^{N}\frac{\partial L_n}{\partial \ln(\bar{z}_{it})}\mathbb{E}_t\hat{z}_{it}\\
+\; & \sum_{i,m}^{N}\frac{1}{2}\frac{\partial^2 L_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{w}_{mt})}\mathbb{E}_t\hat{w}_{it}\hat{w}_{mt}+\sum_{i,m}^{N}\frac{1}{2}\frac{\partial^2 L_n}{\partial \ln(\bar{\kappa}_{nit})\partial \ln(\bar{\kappa}_{nmt})}\mathbb{E}_t\hat{\kappa}_{nit}\hat{\kappa}_{nmt}+\sum_{i,m}^{N}\frac{1}{2}\frac{\partial^2 L_n}{\partial \ln(\bar{z}_{it})\partial \ln(\bar{z}_{mt})}\mathbb{E}_t\hat{z}_{it}\hat{z}_{mt}\\
+\; & \sum_{i,m}^{N}\frac{\partial^2 L_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{\kappa}_{nmt})}\mathbb{E}_t\hat{w}_{it}\hat{\kappa}_{nmt}+\sum_{i,m}^{N}\frac{\partial^2 L_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{z}_{mt})}\mathbb{E}_t\hat{w}_{it}\hat{z}_{mt}+\sum_{i,m}^{N}\frac{\partial^2 L_n}{\partial \ln(\bar{\kappa}_{nit})\partial \ln(\bar{z}_{mt})}\mathbb{E}_t\hat{\kappa}_{nit}\hat{z}_{mt},
\end{aligned}
\tag{C.11}
$$

where

$$
\begin{aligned}
\frac{\partial L_n}{\partial \ln(\bar{w}_{it})} &= \bar{\lambda}_{nit}, \quad \frac{\partial L_n}{\partial \ln(\bar{\kappa}_{nit})} = \bar{\lambda}_{nit}, \quad \frac{\partial L_n}{\partial \ln(\bar{z}_{nt})} = -\frac{1}{\theta}\bar{\lambda}_{nit}, \\
\frac{\partial^2 L_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{w}_{mt})} &= \mathbb{1}(i=m)(-\theta)\bar{\lambda}_{nit} + \theta\bar{\lambda}_{nit}\bar{\lambda}_{nmt}, \\
\frac{\partial^2 L_n}{\partial \ln(\bar{\kappa}_{nit})\partial \ln(\bar{\kappa}_{nmt})} &= \mathbb{1}(i=m)(-\theta)\bar{\lambda}_{nit} + \theta\bar{\lambda}_{nit}\bar{\lambda}_{nmt}, \\
\frac{\partial^2 L_n}{\partial \ln(\bar{z}_{it})\partial \ln(\bar{z}_{mt})} &= -\frac{1}{\theta}\mathbb{1}(i=m)\bar{\lambda}_{nmt} + \left(\frac{1}{\theta}\right)\bar{\lambda}_{nit}\bar{\lambda}_{nmt}, \\
\frac{\partial^2 L_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{\kappa}_{nmt})} &= \mathbb{1}(i=m)(-\theta)\bar{\lambda}_{nit} + \theta\bar{\lambda}_{nit}\bar{\lambda}_{nmt}, \\
\frac{\partial^2 L_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{z}_{mt})} &= \mathbb{1}(m=i)\bar{\lambda}_{nmt} - \bar{\lambda}_{nit}\bar{\lambda}_{nmt}, \text{ and} \\
\frac{\partial^2 L_n}{\partial \ln(\bar{\kappa}_{nit})\partial \ln(\bar{z}_{mt})} &= \mathbb{1}(m=i)\bar{\lambda}_{nmt} - \bar{\lambda}_{nit}\bar{\lambda}_{nmt}.
\end{aligned}
$$

Second-order approximation to labor market clearing

Let $\ln\left(w_{nt}l_{nt}\right) = \ln\left(\sum_{i=1}^{N}\lambda_{int}w_{it}l_{it}\right) \equiv M_n(\lambda_{nt}, w_t, l_t)$. Then, a second-order approximation of $\mathbb{E}_t\hat{w}_{nt} + \mathbb{E}_t\hat{l}_{nt}$ is:

$$
\begin{aligned}
\mathbb{E}_t\hat{w}_{nt} + \mathbb{E}_t\hat{l}_{nt} \approx \quad & \sum_{i=1}^{N}\frac{\partial M_n}{\partial \ln(\bar{\lambda}_{int})}\mathbb{E}_t\hat{\lambda}_{int} + \sum_{i=1}^{N}\frac{\partial M_n}{\partial \ln(\bar{w}_{it})}\mathbb{E}_t\hat{w}_{it} + \sum_{i=1}^{N}\frac{\partial M_n}{\partial \ln(\bar{l}_{it})}\mathbb{E}_t\hat{l}_{it} \\
& + \sum_{i,m}^{N}\frac{1}{2}\frac{\partial^2 M_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{w}_{mt})}\mathbb{E}_t\hat{w}_{it}\hat{w}_{mt} + \sum_{i,m}^{N}\frac{1}{2}\frac{\partial^2 M_n}{\partial \ln(\bar{\lambda}_{int})\partial \ln(\bar{\lambda}_{mnt})}\mathbb{E}_t\hat{\lambda}_{int}\hat{\lambda}_{mnt} \\
& + \sum_{i,m}^{N}\frac{1}{2}\frac{\partial^2 M_n}{\partial \ln(\bar{l}_{it})\partial \ln(\bar{l}_{mt})}\mathbb{E}_t\hat{l}_{it}\hat{l}_{mt} \\
& + \sum_{i,m}^{N}\frac{\partial^2 M_n}{\partial \ln(\bar{\lambda}_{int})\partial \ln(\bar{w}_{mt})}\mathbb{E}_t\hat{\lambda}_{int}\hat{w}_{mt} + \sum_{i,m}^{N}\frac{\partial^2 M_n}{\partial \ln(\bar{\lambda}_{int})\partial \ln(\bar{l}_{mt})}\mathbb{E}_t\hat{\lambda}_{int}\hat{l}_{mt} \\
& + \sum_{i,m}^{N}\frac{\partial^2 M_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{l}_{mt})}\mathbb{E}_t\hat{w}_{it}\hat{l}_{mt}, \qquad \text{(C.12)}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial M_n}{\partial \ln(\bar{\lambda}_{int})} = \frac{\partial M_n}{\partial \ln(\bar{w}_{it})} = \frac{\partial M_n}{\partial \ln(\bar{l}_{it})} &= \frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{\bar{w}_{nt}\bar{l}_{nt}}, \\
\frac{\partial^2 M_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{w}_{mt})} &= 1(m=i)\frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{(\bar{w}_{nt}\bar{l}_{nt})} - \frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}\bar{\lambda}_{mnt}\bar{w}_{mt}\bar{l}_{mt}}{(\bar{w}_{nt}\bar{l}_{nt})^2}, \\
\frac{\partial^2 M_n}{\partial \ln(\bar{\lambda}_{int})\partial \ln(\bar{\lambda}_{mnt})} &= 1(m=i)\frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{(\bar{w}_{nt}\bar{l}_{nt})} - \frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}\bar{\lambda}_{mnt}\bar{w}_{mt}\bar{l}_{mt}}{(\bar{w}_{nt}\bar{l}_{nt})^2}, \\
\frac{\partial^2 M_n}{\partial \ln(\bar{l}_{it})\partial \ln(\bar{l}_{mt})} &= 1(m=i)\frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{(\bar{w}_{nt}\bar{l}_{nt})} - \frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}\bar{\lambda}_{mnt}\bar{w}_{mt}\bar{l}_{mt}}{(\bar{w}_{nt}\bar{l}_{nt})^2}, \\
\frac{\partial^2 M_n}{\partial \ln(\bar{\lambda}_{int})\partial \ln(\bar{w}_{mt})} &= 1(m=i)\frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{(\bar{w}_{nt}\bar{l}_{nt})} - \frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}\bar{\lambda}_{mnt}\bar{w}_{mt}\bar{l}_{mt}}{(\bar{w}_{nt}\bar{l}_{nt})^2}, \\
\frac{\partial^2 M_n}{\partial \ln(\bar{\lambda}_{int})\partial \ln(\bar{l}_{mt})} &= 1(m=i)\frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{(\bar{w}_{nt}\bar{l}_{nt})} - \frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}\bar{\lambda}_{mnt}\bar{w}_{mt}\bar{l}_{mt}}{(\bar{w}_{nt}\bar{l}_{nt})^2}, \text{ and} \\
\frac{\partial^2 M_n}{\partial \ln(\bar{w}_{it})\partial \ln(\bar{l}_{mt})} &= 1(m=i)\frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{(\bar{w}_{nt}\bar{l}_{nt})} - \frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}\bar{\lambda}_{mnt}\bar{w}_{mt}\bar{l}_{mt}}{(\bar{w}_{nt}\bar{l}_{nt})^2}.
\end{aligned}
$$

Appendix D The Multi-Sector Model with Intermediate Inputs

In this section, we derive the equilibrium conditions for the multi-sector model with intermediate inputs used in the quantitative applications, and the corresponding first- and second-order derivatives. Throughout the section, we denote by N the number of locations, indexed by $\{n, i, m, o, h\}$, and by J the number of sectors, indexed by $\{j, k, s\}$.

D.1 Definition of Equilibrium

After taking an expectation over the idiosyncratic shock, the value of a location-sector nj is:

$$
v_t^{nj}(z^t) = U\left(w_t^{nj}(z^t)/P_t^n(z^t)\right) + \nu \ln\left[\sum_{i=1}^{N}\sum_{k=1}^{J} \exp\left(\beta \mathbb{E}[v_{t+1}^{ik}(z^{t+1})|z^t] - m^{nj,ik}\right)^{1/\nu}\right] \tag{D.1}
$$

where P_t^n is the ideal price index in region n in period t as defined below. The fraction of households that relocate from market nj to ik is given by

$$
\mu_t^{nj,ik}(z^t) = \frac{\exp\left(\beta \mathbb{E}[v_{t+1}^{ik}(z^{t+1})|z^t] - m_t^{nj,ik}\right)^{1/\nu}}{\sum_{m=1}^{N}\sum_{h=1}^{J} \exp\left(\beta \mathbb{E}[v_{t+1}^{mh}(z^{t+1})|z^t] - m_t^{nj,mh}\right)^{1/\nu}} \tag{D.2}
$$

The mass of labor in region-sector nj in time $t+1$, l_{t+1}^{nj} evolves according to

$$
l_{t+1}^{nj}(z^t) = \sum_{i=1}^{N}\sum_{k=1}^{J} \mu_t^{ik,nj}(z^t) l_t^{ik}(z^{t-1}) \tag{D.3}
$$

Expenditure share in region nj on goods j from region i at time t is given by

$$
\lambda_t^{nj,ij}(z^t) = \left(\frac{(w_t^{ij}(z^t))^{\gamma^{ij}}(P_t^{ij}(z^t))^{1-\gamma^{ij}}\kappa_t^{nj,ij}}{P_t^{nj}(z^t)}\right)^{-\theta^j} z_t^{ij} \tag{D.4}
$$

where γ^{ij} is the share of value added in gross output.
The price of the sectoral aggregate good j in region n at time t is

$$P_t^{nj}(z^t) = \left[\sum_{i=1}^{N} \left((w_t^{ij}(z^t))^{\gamma^{ij}} (P_t^{ij}(z^t))^{1-\gamma^{ij}} \kappa_t^{nj,ij} \right)^{-\theta^j} z_t^{ij} \right]^{-1/\theta^j}. \tag{D.5}$$

The ideal price index in region n at time t, P_t^n, is defined as

$$P_t^n \equiv \prod_{k=1}^{J} (P_t^{nk}/\alpha^{nk})^{\alpha^{nk}}, \tag{D.6}$$

where α^{nk} is the final expenditure share in sector k and region n with $\sum_k \alpha^{nk} = 1$.
Goods market clearing condition implies the total expenditure on sector j in location n at time t, X_t^{nj}, should satisfy

$$X_t^{ij}(z^t) = (1-\gamma^{ij}) \sum_{n=1}^{N} \lambda_t^{nj,ij}(z^t) X_t^{nj}(z^t) + \alpha^{ij} \sum_k w_t^{ik}(z^t) l_t^{ik}(z^t). \tag{D.7}$$

Finally, labor market clearing is given by

$$w_t^{ij}(z^t) l_t^{ij}(z^t) = \gamma^{ij} \sum_{n=1}^{N} \lambda_t^{nj,ij}(z^t) X_t^{nj}(z^t) \tag{D.8}$$

We now define the stochastic sequential equilibrium of the economy with multi-sector.

Definition 1. *A **stochastic sequential equilibrium** is a set of state-contingent prices* $\left\{ w_t^{nj}(z^t), P_t^{nj}(z^t) \right\}_{n=1,j=1,t=1}^{N,J,T}$, *allocations of goods and labor* $\left\{ \lambda_t^{nj,ij}(z^t), \mu_t^{nj,ik}(z^t), l_t^{nj}(z^{t-1}) \right\}_{n=1,i=1,j=1,k=1,t=1}^{N,N,J,J,T}$, *and the value of locations* $\{ v_{nt}(z^t) \}_{n=1,t=1}^{N,T}$ *that satisfies the equilibrium conditions determined by the location value function (D.1), the gross flows equation (D.2), the law of motion of labor (D.3), the bilateral trade shares (D.4), the local prices ((D.5) and (D.6)), the total expenditures (D.7), and the labor market clearing condition (D.8), and are consistent with beliefs* $f(z_{t+1}|z^t)$.

D.2 Definition of Equilibrium under Heterogeneous Beliefs

Now consider two types of agents, type A and type B, which differ in belief in future productivity. The value of location is type specific. For each $g \in \{A, B\}$,

$$v_t^{nj,g}(z^t) = U\left(w_t^{nj}(z^t)/P_t^n(z^t) \right) + \nu \ln \left[\sum_{i=1}^{N} \sum_{k=1}^{J} \exp\left(\beta \mathbb{E}^g [v_{t+1}^{ik,g}(z^{t+1})|z^t] - m^{nj,ik} \right)^{1/\nu} \right] \tag{D.9}$$

The migration share for each type g is:

$$\mu_t^{nj,ik,g}(z^t) = \frac{\exp\left(\beta \mathbb{E}^g [v_{t+1}^{ik,g}(z^{t+1})|z^t] - m_t^{nj,ik} \right)^{1/\nu}}{\sum_{m=1}^{N} \sum_{h=1}^{J} \exp\left(\beta \mathbb{E}^g [v_{t+1}^{mh,g}(z^{t+1})|z^t] - m_t^{nj,mh} \right)^{1/\nu}} \tag{D.10}$$

The labor allocation in region n sector j evolves according to

$$l_{t+1}^{nj,g}(z^t) = \sum_{i=1}^{N}\sum_{k=1}^{J}\mu_t^{ik,nj,g}(z^t)l_t^{ik,g}(z^{t-1}) \tag{D.11}$$

$$l_{t+1}^{nj}(z^t) = \sum_{g\in\{A,B\}} l_{t+1}^{nj,g}(z^t) \tag{D.12}$$

We now define the stochastic sequential equilibrium in this environment.

Definition 2. ***A stochastic sequential equilibrium with heterogeneous beliefs*** *is a set of state-contingent prices* $\left\{w_t^{nj}(z^t), P_t^{nj}(z^t)\right\}_{n=1,j=1,t=1}^{N,J,T}$*, location values* $\left\{v_t^{nj,g}(z^t)\right\}_{n=1,j=1,t=1,g\epsilon\{A,B\}}^{N,J,T}$*, and allocations of goods and labor* $\left\{\lambda_t^{nj,ij}(z^t), \mu_t^{nj,ik,g}(z^t), l_t^{nj,g}(z^{t-1})\right\}_{n=1,i=1,j=1,k=1,t=1,g\epsilon\{A,B\}}^{N,N,J,J,T}$ *such that* $w_t(z^t), P_t(z^t), \lambda_t, X_t(z^t)$ *solve the trade equilibrium given by the bilateral trade shares (D.4), the local prices ((D.5) and(D.6)), the total expenditures (D.7), and the labor market clearing condition (D.8), and* $v_t^{nj,g}(z^t), \mu_t^{nj,ik,g}(z^t), l_t^g(z^{t-1})$ *solve the dynamic migration decisions given by value function (D.9) for each group* g*, the gross flows equation (D.10), the law of motion of labor (D.11 and D.12), and are consistent with beliefs* $f^g(z_{t+1}|z^t)$*.*

D.3 First-Order Approximations of the Multi-Sector Model

In this subsection, we summarize the first-order approximation of the key equations (D.1)-(D.8) for the multi-sector model. Additional algebra are delegated to Section D.6.1. Throughout, variables with bar are level variables.

The first-order approximation of equations (D.1)-(D.8) is given by:

$$\hat{v}_t^{nj} = \frac{\partial U(\bar{w}_t^{nj}, \bar{P}_t^n)}{\partial \ln(\bar{w}_t^{nj})}\hat{w}_t^{nj} + \frac{\partial U(\bar{w}_t^{nj}, \bar{P}_t^n)}{\partial \ln(\bar{P}_t^n)}\hat{P}_t^n + \beta\sum_{i=1}^{N}\sum_{k=1}^{J}\bar{\mu}_t^{nj,ik}\mathbb{E}_t\hat{v}_{t+1}^{ik} \tag{D.13}$$

$$\hat{\mu}_t^{nj,ik} = \frac{\beta}{\nu}\hat{v}_{t+1}^{ik} - \frac{\beta}{v}\sum_{m=1}^{N}\sum_{h=1}^{J}\bar{\mu}_t^{nj,mh}\mathbb{E}_t\hat{v}_{t+1}^{mh} \tag{D.14}$$

$$\hat{l}_{t+1}^{nj} = \sum_{i=1}^{N}\sum_{k=1}^{J}\frac{\bar{\mu}_t^{ik,nj}\bar{l}_t^{ik}}{\bar{l}_{t+1}^{nj}}\left(\hat{\mu}_t^{ik,nj} + \hat{l}_t^{ik}\right) \tag{D.15}$$

$$\hat{\lambda}_t^{nj,ij} = -\theta^j\left(\gamma^{ij}\hat{w}_t^{ij} + \left(1-\gamma^{ij}\right)\hat{P}_t^{ij} - \hat{P}_t^{nj} + \hat{\kappa}_t^{nj,ij}\right) + \hat{z}_t^{ij} \tag{D.16}$$

$$\hat{P}_t^{nj} = \sum_{i=1}^{N}\bar{\lambda}_t^{nj,ij}\left(\gamma^{ij}\hat{w}_t^{ij} + \left(1-\gamma^{ij}\right)\hat{P}_t^{ij} + \hat{\kappa}_t^{nj,ij} - \frac{1}{\theta^j}\hat{z}_t^{ij}\right) \tag{D.17}$$

$$\hat{P}_t^n = \sum_{j=1}^{J}\alpha^{nj}\hat{P}_t^{nj} \tag{D.18}$$

$$\hat{X}_t^{ij} = (1-\gamma^{ij})\sum_{n}^{N}\bar{\varrho}_t^{nj,ij}\left(\hat{\lambda}_t^{nj,ij} + \hat{X}_t^{nj}\right) + \alpha^{ij}\sum_{k}\bar{\zeta}_t^{ik,ij}\left(\hat{w}_t^{ik} + \hat{l}_t^{ik}\right) \tag{D.19}$$

$$\hat{w}_t^{ij} + \hat{l}_t^{ij} = \gamma^{ij}\sum_{n}^{N}\bar{\chi}_t^{nj,ij}\left(\hat{\lambda}_t^{nj,ij} + \hat{X}_t^{nj}\right). \tag{D.20}$$

And for $t' > t$, (D.21)

$$\mathbb{E}_t\hat{v}^{nj}_{t'} = \frac{\partial U(\bar{w}^{nj}_{t'},\bar{P}^n_{t'})}{\partial \ln(\bar{w}^{nj}_{t'})}\mathbb{E}_t\hat{w}^{nj}_{t'} + \frac{\partial U(\bar{w}^{nj}_{t'},\bar{P}^n_{t'})}{\partial \ln(\bar{P}^n_{t'})}\mathbb{E}_t\hat{P}^n_n + \sum_{i=1}^{N}\sum_{k=1}^{J}\bar{\mu}^{nj,ik}_t\beta\mathbb{E}_t\hat{v}^{ik}_{t'+1} \tag{D.22}$$

$$\mathbb{E}_t\hat{\mu}^{nj,ik}_{t'} = \frac{\beta}{\nu}\mathbb{E}_t\hat{v}^{ik}_{t'+1} - \frac{\beta}{v}\sum_{m=1}^{N}\sum_{h=1}^{J}\bar{\mu}^{nj,mh}_t\mathbb{E}_t\hat{v}^{mh}_{t'+1} \tag{D.23}$$

$$\mathbb{E}_t\hat{l}^{nj}_{t'+1} = \sum_{i=1}^{N}\sum_{k=1}^{J}\frac{\bar{\mu}^{ik,nj}_{t'}\bar{l}^{ik}_{t'}}{\bar{l}^{nj}_{t'+1}}\left(\mathbb{E}_t\hat{\mu}^{ik,nj}_{t'} + \mathbb{E}_t\hat{l}^{ik}_{t'}\right) \tag{D.24}$$

$$\mathbb{E}_t\hat{\lambda}^{nj,ij}_{t'} = -\theta^j\left(\gamma^{ij}\mathbb{E}_t\hat{w}^{ij}_{t'} + \left(1-\gamma^{ij}\right)\mathbb{E}_t\hat{P}^{ij}_{t'} - \mathbb{E}_t\hat{P}^{nj}_{t'} + \mathbb{E}_t\hat{\kappa}^{nj,ij}_{t'}\right) + \mathbb{E}_t\hat{z}^{ij}_{t'} \tag{D.25}$$

$$\mathbb{E}_t\hat{P}^{nj}_{t'} = \sum_{i=1}^{N}\bar{\lambda}^{nj,ij}_{t'}\left(\gamma^{ij}\mathbb{E}_t\hat{w}^{ij}_{t'} + \left(1-\gamma^{ij}\right)\mathbb{E}_t\hat{P}^{ij}_{t'} + \mathbb{E}_t\hat{\kappa}^{nj,ij}_{t'} - \frac{1}{\theta^j}\mathbb{E}_t\hat{z}^{ij}_{t'}\right) \tag{D.26}$$

$$\mathbb{E}_t\hat{P}^{n}_{t'} = \sum_{j=1}^{J}\alpha^{nj}\mathbb{E}_t\hat{P}^{nj}_{t'} \tag{D.27}$$

$$\mathbb{E}_t\hat{X}^{ij}_{t'} = (1-\gamma^{ij})\sum_{n}^{N}\bar{\varrho}^{nj,ij}_{t'}\left(\mathbb{E}_t\hat{\lambda}^{nj,ij}_{t'} + \mathbb{E}_t\hat{X}^{nj}_{t'}\right) + \alpha^{ij}\sum_{k}\bar{\zeta}^{ik,ij}_{t'}\left(\mathbb{E}_t\hat{w}^{ik}_{t'} + \mathbb{E}_t\hat{l}^{ik}_{t'}\right) \tag{D.28}$$

$$\mathbb{E}_t\hat{w}^{ij}_{t'} + \mathbb{E}_t\hat{l}^{ij}_{t'} = \gamma^{ij}\sum_{n}^{N}\bar{\chi}^{nj,ij}_{t'}\left(\mathbb{E}_t\hat{\lambda}^{nj,ij}_{t'} + \mathbb{E}_t\hat{X}^{nj}_{t'}\right). \tag{D.29}$$

Approximation Points (level values):

$$\bar{\psi}^{ik,nj}_{t+1} = \frac{\bar{\mu}^{ik,nj}_t\bar{l}^{ik}_t}{\bar{l}^{nj}_{t+1}} \tag{D.30}$$

$$\bar{\varrho}^{nj,ij}_t = \frac{\bar{\lambda}^{nj,ij}_t\bar{X}^{nj}_t}{\bar{X}^{ij}_t} \tag{D.31}$$

$$\bar{\zeta}^{ik,ij}_t = \frac{\bar{w}^{ik}_t\bar{l}^{ik}_t}{\bar{X}^{ij}_t} \tag{D.32}$$

$$\bar{\chi}^{nj,ij}_t = \frac{\bar{\lambda}^{nj,ij}_t\bar{X}^{nj}_t}{\bar{w}^{ij}_t\bar{l}^{ij}_t}, \tag{D.33}$$

where $\bar{\lambda}^{nj,ij}_t$ is defined in (D.4) and $\bar{\mu}^{ik,nj}_t$ is defined in (D.2)

D.4 Second-Order Approximations of the Multi-Sector Model

In this subsection, we summarize the second-order approximation of the key equations (D.1)-(D.8) for the multi-sector model. Additional algebra are delegated to Section D.6.2. Throughout, variables with bar are level variables. To shorten notations, for approximating migration and labor market flow equations in the system of equations (D.34), we *combine the region and sector notations*, using superscripts $\{n,i,m,o\}_1^{N\times J}$ to refer to pair of region-sector.

$$
\begin{aligned}
\mathbb{E}_t \hat{v}^n_{t+1} &= \frac{\partial U(\bar{w}^n_{t+1}, \bar{P}^n_{t+1})}{\partial \ln(\bar{w}^n_{t+1})}\mathbb{E}_t \hat{w}^n_{t+1} + \frac{\partial U(\bar{w}^n_{t+1}, \bar{P}^n_{t+1})}{\partial \ln(\bar{P}^n_{t+1})}\mathbb{E}_t \hat{P}^n_{t+1} \\
&+ \frac{1}{2}\frac{\partial^2 U(\bar{w}^n_{t+1}, \bar{P}^n_{t+1})}{\partial \ln(\bar{w}^n_{t+1})^2}\mathbb{E}_t(\hat{w}^n_{t+1})^2 + \frac{1}{2}\frac{\partial^2 U(\bar{w}^n_{t+1}, \bar{P}^n_{t+1})}{ln(\partial \bar{P}^n_{t+1})^2}\mathbb{E}_t(\hat{P}^n_{t+1})^2 + \frac{\partial^2 U(\bar{w}^n_{t+1}, \bar{P}^n_{t+1})}{\partial \ln(\bar{w}^n_{t+1})\partial \ln(\bar{P}^n_{t+1})}\mathbb{E}_t \hat{w}^n_{t+1}\hat{P}^n_{t+1} \\
&+ \sum_{o}^{N\times J} \beta \bar{\mu}^{no}_{t+1}\mathbb{E}_t \hat{v}^o_{t+2} + \frac{1}{2}\sum_{m}^{N\times J}\sum_{o}^{N\times J}\frac{\beta^2}{\nu}\bar{\mu}^{nm}_{t+1}\Big(\mathbb{1}(o=m) - \bar{\mu}^{no}_{t+1}\Big)\mathbb{E}_t \hat{v}^m_{t+2}\hat{v}^o_{t+2} \\
\mathbb{E}_t \hat{\mu}^{ni}_t &= \sum_{o}^{N\times J}\frac{\beta}{\nu}\Big(\mathbb{1}(o=i) - \bar{\mu}^{no}_t\Big)\mathbb{E}_t \hat{v}^o_{t+1} \\
&+ \frac{1}{2}\sum_{o,m}^{N\times J}(\frac{\beta}{\nu})^2\Big[\mathbb{1}(o=i)\bar{\mu}^{ni}_t(\mathbb{1}(m=i) - \bar{\mu}^{nm}_t) - \bar{\mu}^{no}_t\bar{\mu}^{ni}_t[\mathbb{1}(m=i) - 2\bar{\mu}^{nm}_t + \mathbb{1}(m=o)]\Big]\mathbb{E}_t \hat{v}^o_{t+1}\hat{v}^m_{t+1} \\
\mathbb{E}_t \hat{l}^n_{t+1} &= \sum_{o=1}^{N\times J}\sum_{i=1}^{N\times J}\frac{\beta}{\nu}\bar{\psi}^{in}_{t+1}\Big(\mathbb{1}(o=n) - \bar{\mu}^{io}_t\Big)\mathbb{E}_t \hat{v}^o_{t+1} + \sum_{o=1}^{N\times J}\bar{\psi}^{on}_{t+1}\mathbb{E}_t \hat{l}^o_t \\
&+ \frac{1}{2}\sum_{o,m}^{N\times J}(\frac{\beta}{\nu})^2\Big[-\sum_{i=1}^{N}\bar{\mu}^{io}_t\bar{\psi}^{in}_{t+1}[\mathbb{1}(m=n) + \mathbb{1}(m=o) - 2\bar{\mu}^{im}_t] + \mathbb{1}(o=n)\sum_i \bar{\psi}^{in}_{t+1}(\mathbb{1}(m=n) - \bar{\mu}^{im}_t) \\
&- [\sum_{i=1}^{N\times J}\bar{\psi}^{in}_{t+1}(\mathbb{1}(o=n) - \bar{\mu}^{io}_t)][\sum_{i}^{N\times J}\bar{\psi}^{in}_{t+1}(\mathbb{1}(m=n) - \bar{\mu}^{im}_t)]\Big]\mathbb{E}_t \hat{v}^o_{t+1}\hat{v}^m_{t+1} \\
&+ \frac{1}{2}\sum_{o,m}^{N\times J}[\mathbb{1}(o=m)\bar{\psi}^{mn}_{t+1} - \bar{\psi}^{on}_{t+1}\bar{\psi}^{mn}_{t+1}]\,\mathbb{E}_t \hat{l}^o_t \hat{l}^m_t \\
&+ \sum_{o,m}^{N\times J}\left[\frac{\beta}{\nu}\bar{\psi}^{mn}_{t+1}(\mathbb{1}(o=n) - \bar{\mu}^{mo}_t) - \frac{\beta}{\nu}\bar{\psi}^{mn}_{t+1}[\sum_{i}^{N\times J}\bar{\psi}^{in}_{t+1}(\mathbb{1}(o=n) - \bar{\mu}_{iot})]\right]\mathbb{E}_t \hat{v}^o_{t+1}\hat{l}^m_t, \qquad \text{(D.34)}
\end{aligned}
$$

Trade equilibrium conditions:

$$
\begin{aligned}
\mathbb{E}_t\hat{\pi}_t^{nj,ij} &= -\theta^j\left(\gamma^{ij}\mathbb{E}_t\hat{w}_t^{ij} + \left(1-\gamma^{ij}\right)\mathbb{E}_t\hat{P}_t^{ij} - \mathbb{E}_t\hat{P}_t^{nj} + \mathbb{E}_t\hat{\kappa}_t^{nj,ij}\right) + \mathbb{E}_t\hat{z}_t^{ij},\\
\mathbb{E}_t\hat{P}_t^{nj} &= \sum_{k=1}^{N}\bar{\pi}_t^{nj,kj}\left(\gamma^{kj}\mathbb{E}_t\hat{w}_t^{kj} + \left(1-\gamma^{kj}\right)\mathbb{E}_t\hat{P}_t^{kj} + \mathbb{E}_t\hat{\kappa}_t^{nj,kj} - \frac{1}{\theta^j}\mathbb{E}_t\hat{z}_t^{kj}\right)\\
&+ \frac{1}{2}\sum_{k,m}^{N}(-\theta^j)\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k)(\gamma^{kj})^2 - \gamma^{kj}\gamma^{mj}\bar{\lambda}_t^{nj,kj}\Big)\mathbb{E}_t\hat{w}_t^{kj}\hat{w}_t^{mj}\\
&+ \frac{1}{2}\sum_{k,m}^{N}(-\theta^j)\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k)(1-\gamma^{kj})^2 - (1-\gamma^{kj})(1-\gamma^{mj})\bar{\lambda}_t^{nj,kj}\Big)\mathbb{E}_t\hat{P}_t^{kj}\hat{P}_t^{mj}\\
&+ \frac{1}{2}\sum_{k,m}^{N}\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big)\mathbb{E}_t\Big((-\theta^j)\hat{\kappa}_t^{nj,kj}\hat{\kappa}_t^{nj,mj} + \Big(-\frac{1}{\theta^j}\Big)\mathbb{E}_t\hat{z}_t^{kj}\hat{z}_t^{mj} + 2\mathbb{E}_t\hat{\kappa}_t^{nj,kj}\hat{z}_t^{mj}\Big)\\
&+ \sum_{k,m}^{N}(-\theta^j)\gamma^{kj}(1-\gamma^{mj})\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big)\mathbb{E}_t\hat{w}_t^{kj}\hat{P}_t^{mj}\\
&+ \sum_{k,m}^{N}(1-\gamma^{kj})\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big)\Big(\mathbb{E}_t\hat{P}_t^{kj}\hat{z}_t^{mj} + (-\theta^j)\mathbb{E}_t\hat{P}_t^{kj}\hat{\kappa}_t^{nj,mj}\Big)\\
&+ \sum_{k,m}^{N}\gamma^{kj}\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big)\Big((-\theta^j)\mathbb{E}_t\hat{w}_t^{kj}\hat{\kappa}_t^{nj,mj} + \mathbb{E}_t\hat{w}_t^{kj}\hat{z}_t^{mj}\Big),\\
\mathbb{E}_t\hat{P}_t^{n} &= \sum_{j=1}^{J}\alpha^{nj}\mathbb{E}_t\hat{P}_t^{nj},\\
\mathbb{E}_t\hat{X}_t^{ij} &= (1-\gamma^{ij})\sum_{n}^{N}\bar{\varrho}_t^{nj,ij}\left(\mathbb{E}_t\hat{\lambda}_t^{nj,ij} + \mathbb{E}_t\hat{X}_t^{nj}\right) + \alpha^{ij}\sum_{k}\bar{\zeta}_t^{ik,ij}\left(\mathbb{E}_t\hat{w}_t^{ik} + \mathbb{E}_t\hat{l}_t^{ik}\right)\\
&+ \frac{1}{2}\sum_{n,m}^{N}(1-\gamma^{ij})\bar{\varrho}_t^{mj,ij}\Big(\mathbb{1}(m=n) - (1-\gamma^{ij})\bar{\varrho}_t^{nj,ij}\Big)\Big(\mathbb{E}_t\hat{\lambda}_t^{nj,ij}\hat{\lambda}_t^{mj,ij} + \mathbb{E}_t\hat{X}_t^{nj}\hat{X}_t^{mj} + 2\mathbb{E}_t\hat{\lambda}_t^{nj,ij}\hat{X}_t^{mj}\Big)\\
&+ \frac{1}{2}\sum_{k,s}^{J}\alpha^{ij}\bar{\zeta}_t^{ik,ij}\Big(\mathbb{1}(k=s) - \alpha^{ij}\bar{\zeta}_t^{is,ij}\Big)\Big(\mathbb{E}_t\hat{w}_t^{ik}\hat{w}_t^{is} + \mathbb{E}_t\hat{l}_t^{ik}\hat{l}_t^{is} + 2\mathbb{E}_t\hat{w}_t^{ik}\hat{l}_t^{is}\Big)\\
&+ \sum_{n}^{N}\sum_{s}^{J}\Big(-(1-\gamma^{ij})\alpha^{ij}\bar{\varrho}_t^{njij}\bar{\zeta}_t^{is,ij}\Big)\Big(\mathbb{E}_t\hat{\lambda}_t^{nj,ij}\hat{w}_t^{ij} + \mathbb{E}_t\hat{\lambda}_t^{nj}\hat{l}_t^{is} + \mathbb{E}_t\hat{X}_t^{nj}\hat{w}_t^{ij} + \mathbb{E}_t\hat{X}_t^{nj}\hat{l}_t^{is}\Big), \text{ and}\\
\mathbb{E}_t\hat{w}_t^{ij} &= \gamma^{ij}\sum_{n=1}^{N}\bar{\chi}_t^{nj,ij}\left(\mathbb{E}_t\hat{\lambda}_t^{nj,ij} + \mathbb{E}_t\hat{X}_t^{nj}\right) - \mathbb{E}_t\hat{l}_t^{ij}\\
&+\frac{1}{2}\sum_{n,m}^{N}\gamma^{ij}\bar{\chi}_t^{nj,ij}\Big(\mathbb{1}(m=n) - \gamma^{ij}\bar{\chi}_t^{mj,ij}\Big)\Big(\mathbb{E}_t\hat{\lambda}_t^{nj,ij}\hat{\lambda}_t^{mj,ij} + \mathbb{E}_t\hat{X}_t^{nj}\hat{X}_t^{mj} + 2\mathbb{E}_t\hat{\lambda}_t^{nj,ij}\hat{X}_t^{mj}\Big). \quad \text{(D.35)}
\end{aligned}
$$

D.5 Solution Algorithms

D.5.1 Solving for Future Outcomes according to Belief in t

Let the history until t be denoted by z^t. The following algorithm solves the decision of agents in period t, $\hat{\mu}_t^{nj,ik}$, and agents expectation about future outcomes, e.g., $\mathbb{E}_t\mu_{t'}^{nk,ij}$.

i. Initiate the algorithm by guessing a sequence of expected value of locations $\{\mathbb{E}_t\hat{v}_{t'}^{nj(0)}\}_{t'=t+1}^{T}$. The '0' indicates the values will be iteratively updated.

ii. Use $\{\hat{v}_{t'}^{nj(0)}\}_{t'=t+1}^{T}$ to solve for the current period migration decision, $\hat{\mu}_t^{nj,ik}$, and expected future migration decisions $\{\mathbb{E}_t\hat{\mu}_{t'}^{nj,ik}\}_{t'=t+1}^{T}$ using equation (D.23):

$$\begin{aligned}\hat{\mu}_t^{nj,ik}(z^t) &= \frac{\beta}{v}\mathbb{E}_t\hat{v}_{t+1}^{ik} - \frac{\beta}{v}\sum_{m=1}^{N}\sum_{h=1}^{J}\bar{\mu}_t^{nj,mh}\mathbb{E}_t\hat{v}_{t+1}^{mh(0)} \\ \mathbb{E}_t\hat{\mu}_{t'}^{nj,ik} &= \frac{\beta}{v}\mathbb{E}_t\hat{v}_{t'+1}^{ik} - \frac{\beta}{v}\sum_{m=1}^{N}\sum_{h=1}^{J}\bar{\mu}_{t'}^{nj,mh}\mathbb{E}_t\hat{v}_{t'+1}^{mh(0)}.\end{aligned}$$

iii. Use the path for $\hat{\mu}_t(z^t)$ and $\{\mathbb{E}_t\hat{\mu}_{t'}\}_{t'=t+1}^{T}$ and $\hat{l}_t^{nj} = 0$ to get the next period labor, $\hat{l}_{t+1}^{nj}$, and the consecutive expected future labor path $\{\mathbb{E}_t\hat{l}_{t'}^{nj}\}_{t'=t+1}^{T}$ using equation (D.24):

$$\begin{aligned}\hat{l}_{t+1}^{nj}(z^t) &= \sum_{i=1}^{N}\sum_{k=1}^{J}\bar{\psi}_{t+1}^{ik,nj}\left(\hat{\mu}_t^{ik,nj}(z^t) + \hat{l}_t^{ik}(z^{t-1})\right) \\ \mathbb{E}_t\hat{l}_{t'+1}^{nj} &= \sum_{i=1}^{N}\sum_{k=1}^{J}\bar{\psi}_{t'+1}^{ik,nj}\left(\mathbb{E}_t\hat{\mu}_{t'}^{ik,nj} + \mathbb{E}_t\hat{l}_{t'}^{ik}\right).\end{aligned}$$

iv. Solve for the temporary equilibrium:

1. Given $\hat{l}_t^{nj}$, $\{\mathbb{E}_t\hat{l}_{t'+1}^{nj}\}_{t'=t}^{T}$, $\hat{z}_t$, and $\{\mathbb{E}_t\hat{z}_{t'}\}_{t'=t}^{T}$, solve for the current nominal wage, $\hat{w}_t$, and the expected future nominal wage ,$\{\mathbb{E}_t\hat{w}_{t'}\}_{t'=t}^{T}$, using the matrix inversion derived in Section D.6.3 of this appendix for the static trade equilibrium in each period $t' = t, t+1, ..., T$:

$$\hat{w}_{t'} = [\mathbf{I} - \tilde{M}_{t'} - \tilde{T}_{t'}]^{-1}\left((\tilde{T}_{t'} - \mathbf{I})\hat{l}_{t'} + \tilde{Q}_{t'}\hat{z}_{t'} + \tilde{F}_{t'}\hat{K}_{t'}\right), \tag{D.36}$$

in which $\hat{w}_{t'}$ is a vector obtained from stacking the change in wage across locations and sectors; and other variables in these equations are matrices defined in Section D.6.3.

2. Recover $\hat{\lambda}_{t'}$, $\hat{P}_{t'}^{nj}$, $\hat{P}_{t'}^{n}$, $\hat{X}_{t'}$ using equations (D.25), (D.26), (D.27), (D.28), respectively.

v. Use T period real wage to compute the expected value of location-sector nj at time T, $\hat{v}_T^{nj}$, which is given by the flow utility:

$$\mathbb{E}_t\hat{v}_T^{nj(1)} = \frac{\partial U(\bar{w}_T^{nj}, \bar{P}_T^{n})}{\partial \ln(\bar{w}_T^{nj})}\mathbb{E}_t\hat{w}_T^{nj} + \frac{\partial U(\bar{w}_T^{nj}, \bar{P}_T^{n})}{\partial \ln(\bar{P}_T^{n})}\mathbb{E}_t\hat{P}_T^{n}$$

vi. For each period, starting from $\hat{v}_T^{nj(1)}$, solve backward for $\{\mathbb{E}_t\hat{v}_{t'}^{nj(1)}\}_{t'=t+1}^{T-1}$ using

$$\mathbb{E}_t\hat{v}_{t'}^{nj(1)} = \frac{\partial U(\bar{w}_{t'}^{nj}, \bar{P}_{t'}^{n})}{\partial \ln(\bar{w}_{t'}^{nj})}\mathbb{E}_t\hat{w}_{t'}^{nj} + \frac{\partial U(\bar{w}_{t'}^{nj}, \bar{P}_{t'}^{n})}{\partial \ln(\bar{P}_{t'}^{n})}\mathbb{E}_t\hat{P}_{t'}^{n} + \sum_{i=1}^{N}\sum_{k=1}^{J}\bar{\mu}_{t'}^{nj,ik}\beta\mathbb{E}_t\hat{v}_{t'+1}^{ik(1)}$$

vii. This delivers an updated path for $\{\mathbb{E}_t\hat{v}_{t'}^{nj(1)}\}_{t'=t+1}^{T}$.

viii. Check if $\{\mathbb{E}_t\hat{v}_{t'}^{nj(1)}\}_{t'=t+1}^{T} \approx \{\mathbb{E}_t\hat{v}_{t'}^{nj(0)}\}_{t'=t+1}^{T}$.

ix. If not, take the path for $\{\mathbb{E}_t\hat{v}_{t'}^{nj(1)}\}_{t'=t+1}^{T}$ as the new set of initial conditions in step 1 and iterate until step 8 holds.

D.5.2 Solving for the Allocation Sequence for a Given Realization of the Path of Fundamentals

i. Start from period $t = t_0$.

ii. Solve for the first-order approximation for each $t' \geq t$ under the belief $\{\mathbb{E}_t z_{t'}\}_{t'=t}^{T}$ using the algorithm explained in Section D.5.1. Note that at the initial period, $t_0 = 1$, $\hat{l}_1^{nj} = 0$.

iii. Save the 'realized' outcomes; current period's nominal wage ($\hat{w}_t$), price ($\hat{p}_t$), ideal price index ($\hat{P}_t$), trade share ($\hat{\pi}_t$), migration share ($\hat{\mu}_t$), and the next period's labor allocation ($\hat{l}_{t+1}$).

iv. The economy moves to the next period $t = t_0 + 1$. Repeat the above process for $t_0 + 1$ period. Use $\hat{l}_{t_0+1}$ obtained at t_0 as an initial value of labor allocation.

v. Repeat this until $t = T$.

D.6 Additional Derivations

This subsection provides additional algebra for the results in Section D.3 and D.4 of this appendix. We also derive closed-form expressions for first- and second-order counterfactual static trade equilibria, which enables us to solve each static trade equilibrium via a single matrix inversion. We illustrate the approximation to expected values using the case when the variable and the expectation operator are from the same period (e.g., $\mathbb{E}_t \hat{\lambda}_t^{nj,ij}$), in which case the expectation is redundant, but the same approximation holds for expectation of future values (e.g., $\mathbb{E}_t \hat{\lambda}_{t'}^{nj,ij}, t' > t$).

D.6.1 First-Order Derivatives

First-order approximation to trade share

Let $\ln \lambda_t^{nj,ij} = \ln \left[\left(\frac{(w_t^{ij})^{\gamma^{ij}} (P_t^{ij})^{1-\gamma^{ij}} \kappa_t^{nj,ij}}{P_t^{nj}} \right)^{-\theta^j} z_t^{ij} \right] \equiv K(w_t^{ij}, P_t^{ij}, \kappa_t^{nj,ij}, P_t^{nj}, z_t^{ij})$. Then we approximate $\mathbb{E}_t \hat{\lambda}_{nit}$ using

$$\begin{aligned} \mathbb{E}_t \hat{\lambda}_t^{nj,ij} &= \left(\frac{\partial K}{\partial \ln(\bar{w}_t^{ij})} \mathbb{E}_t \hat{w}_t^{ij} + \frac{\partial K}{\partial \ln(\bar{P}_t^{ij})} \mathbb{E}_t \hat{P}_t^{ij} + \frac{\partial K}{\partial \ln(\bar{\kappa}_t^{ij,nj})} \mathbb{E}_t \hat{\kappa}_t^{ijnj} + \frac{\partial K}{\partial \ln(\bar{P}_t^{nj})} \mathbb{E}_t \hat{P}_t^{nj} + \frac{\partial K}{\partial \ln(\bar{z}_t^{ij})} \hat{z}_t^{ij} \right) \\ &= -\theta^j \left(\gamma^{ij} \mathbb{E}_t \hat{w}_t^{ij} + \left(1 - \gamma^{ij}\right) \mathbb{E}_t \hat{P}_t^{ij} - \mathbb{E}_t \hat{P}_t^{nj} + \mathbb{E}_t \hat{\kappa}_t^{nj,ij} \right) + \mathbb{E}_t \hat{z}_t^{ij}. \end{aligned} \tag{D.37}$$

First-order approximation to sectoral price index

Let $\ln P_t^{nj} = \ln \left[\sum_{i=1}^{N} \left((w_t^{ij})^{\gamma^{ij}} (P_t^{ij})^{1-\gamma^{ij}} \kappa_t^{nj,ij} \right)^{-\theta^j} z_t^{ij} \right]^{-1/\theta^j} \equiv L(w_t^j, P_t^j, \kappa_t^{nj}, z_t^j)$ where $\kappa_t^{nj} = [\kappa_t^{nj,1j}, ..., \kappa_t^{nj,Nj}]'$. Then, a first-order approximation of $\mathbb{E}_t \hat{P}_t^{nj}$ is:

$$\begin{aligned} \mathbb{E}_t \hat{P}_t^{nj} &\approx \sum_{i=1}^{N} \left(\frac{\partial L}{\partial \ln(\bar{w}_t^{ij})} \mathbb{E}_t \hat{w}_t^{ij} + \frac{\partial L}{\partial \ln(\bar{P}_t^{ij})} \mathbb{E}_t \hat{P}_t^{ij} + \frac{\partial L}{\partial \ln(\bar{\kappa}_t^{nj,ij})} \mathbb{E}_t \hat{\kappa}_t^{nj,ij} + \frac{\partial L}{\partial \ln(\bar{z}_t^{ij})} \mathbb{E}_t \hat{z}_t^{ij} \right) \\ &= \sum_{i=1}^{N} \bar{\pi}_t^{nj,ij} \left(\gamma^{ij} \mathbb{E}_t \hat{w}_t^{ij} + \left(1 - \gamma^{ij}\right) \mathbb{E}_t \hat{P}_t^{ij} + \mathbb{E}_t \hat{\kappa}_t^{nj,ij} - \frac{1}{\theta^j} \mathbb{E}_t \hat{z}_t^{ij} \right). \end{aligned} \tag{D.38}$$

First-order approximation to ideal price index

Let $\ln P_t^n = \ln \left[\prod_{k=1}^{J} (P_t^{nk} / \alpha^{nk})^{\alpha^{nk}} \right] \equiv Q(P_t^n)$ where $P_t^n = [P_t^{n1}, ..., P_t^{nJ}]'$. Then, a first-order approximation of $\mathbb{E}_t \hat{P}_t^n$ is:

$$\begin{aligned}\mathbb{E}_t \hat{P}_t^n &\approx \prod_{j=1}^{J} \Big(\frac{\partial Q}{\partial \ln(\bar{P}_t^{nj})} \mathbb{E}_t \hat{P}_t^{nj} \Big) \\ &= \sum_{j=1}^{J} \alpha^{nj} \mathbb{E}_t \hat{P}_t^{nj}. \end{aligned} \tag{D.39}$$

First-order approximation to expenditure

Let $\ln\left(X_t^{ij}\right) = \ln\left((1-\gamma^{ij}) \sum_{n=1}^{N} \lambda_t^{nj,ij} X_t^{nj} + \alpha^{ij} \sum_k w_t^{ik} l_t^{ik}\right) \equiv O(\lambda_t^{j,ij}, X_t^j, w_t^i, l_t^i)$ where $\lambda_t^{j,ij} = [\lambda_t^{1j,ij}, ..., \lambda_t^{nj,ij}]'$, $X_t^j = [X_t^{1j}, ..., X_t^{Nj}]$, $w_t^i = [w_t^{i1}, ..., w_t^{iJ}]$, and $l_t^i = [l_t^{i1}, ..., l_t^{iJ}]$. Then, a first-order approximation of $\mathbb{E}_t \hat{X}_t^{ij}$ is:

$$\begin{aligned}\mathbb{E}_t \hat{X}_t^{ij} &\approx \sum_{n=1}^{N} \Big(\frac{\partial O}{\partial \ln(\bar{\lambda}_t^{nj,ij})} \mathbb{E}_t \hat{\lambda}_t^{nj,ij} + \frac{\partial O}{\partial \ln(\bar{X}_t^{nj})} \mathbb{E}_t \hat{X}_t^{nj} \Big) + \sum_{k=1}^{J} \Big(\frac{\partial L}{\partial \ln(\bar{w}_t^{ik})} \mathbb{E}_t \hat{w}_t^{ik} + \frac{\partial O}{\partial \ln(\bar{l}_t^{ik})} \mathbb{E}_t \hat{l}_t^{ik} \Big) \\ &= (1-\gamma^{ij}) \sum_{n}^{N} \bar{\varrho}_t^{nj,ij} \left(\mathbb{E}_t \hat{\lambda}_t^{nj,ij} + \mathbb{E}_t \hat{X}_t^{nj} \right) + \alpha^{ij} \sum_k \bar{\zeta}_t^{ik,ij} \left(\mathbb{E}_t \hat{w}_t^{ik} + \mathbb{E}_t \hat{l}_t^{ik} \right). \end{aligned} \tag{D.40}$$

First-order approximation to labor market clearing

Let $\ln\left(w_t^{ij} l_t^{ij}\right) = \ln\left(\gamma^{ij} \sum_{n=1}^{N} \lambda_t^{nj,ij} X_t^{nj}\right) \equiv R(\lambda_t^{j,ij}, X_t^j)$. Then, a first-order approximation of $\mathbb{E}_t \hat{w}_{it}$ is:

$$\begin{aligned}\mathbb{E}_t \hat{w}_t^{ij} + \mathbb{E}_t \hat{l}_t^{ij} &\approx \sum_{n=1}^{N} \Big(\frac{\partial R}{\partial \ln(\bar{\lambda}_t^{nj,ij})} \mathbb{E}_t \hat{\lambda}_t^{nj,ij} + \frac{\partial R}{\partial \ln(\bar{X}_t^{nj})} \mathbb{E}_t \hat{X}_t^{nj} \Big) \\ &= \gamma^{ij} \sum_{n=1}^{N} \bar{\chi}_t^{nj,ij} \left(\mathbb{E}_t \hat{\lambda}_t^{nj,ij} + \mathbb{E}_t \hat{X}_t^{nj} \right). \end{aligned} \tag{D.41}$$

D.6.2 Second-Order Derivatives

Second-order approximation to trade share

First order approximation for trade share λ is exact. Therefore, a second-order approximation of $\mathbb{E}_t \hat{\lambda}_{nit}$ is:

$$\begin{aligned}\mathbb{E}_t \hat{\lambda}_t^{nj,ij} &= \Big(\frac{\partial K}{\partial \ln(\bar{w}_t^{ij})} \mathbb{E}_t \hat{w}_t^{ij} + \frac{\partial K}{\partial \ln(\bar{P}_t^{ij})} \mathbb{E}_t \hat{P}_t^{ij} + \frac{\partial K}{\partial \ln(\bar{\kappa}_t^{ij,nj})} \mathbb{E}_t \hat{\kappa}_t^{ijnj} + \frac{\partial K}{\partial \ln(\bar{P}_t^{nj})} \mathbb{E}_t \hat{P}_t^{nj} + \frac{\partial K}{\partial \ln(\bar{z}_t^{ij})} \mathbb{E}_t \hat{z}_t^{ij} \Big) \\ &= -\theta^j \left(\gamma^{ij} \mathbb{E}_t \hat{w}_t^{ij} + \left(1-\gamma^{ij}\right) \mathbb{E}_t \hat{P}_t^{ij} - \mathbb{E}_t \hat{P}_t^{nj} + \mathbb{E}_t \hat{\kappa}_t^{nj,ij} \right) + \mathbb{E}_t \hat{z}_t^{ij}. \end{aligned} \tag{D.42}$$

Second-order approximation to sectoral price index

Let $\ln P_t^{nj} = \ln\left[\sum_{i=1}^{N} \left((w_t^{ij})^{\gamma^{ij}} (P_t^{ij})^{1-\gamma^{ij}} \kappa_t^{nj,ij} \right)^{-\theta^j} z_t^{ij} \right]^{-1/\theta^j} \equiv L(w_t^j, P_t^j, \kappa_t^{nj}, z_t^j)$. Then, a second-order approximation of $\mathbb{E}_t \hat{P}_t^{nj}$ is:

$$
\begin{aligned}
\mathbb{E}_t \hat{P}_t^{nj} \approx & \sum_{i=1}^{N} \frac{\partial L}{\partial \ln(\bar{w}_t^{ij})} \mathbb{E}_t \hat{w}_t^{ij} + \sum_{i=1}^{N} \frac{\partial L}{\partial \ln(\bar{P}_t^{ij})} \mathbb{E}_t \hat{P}_t^{ij} + \sum_{i=1}^{N} \frac{\partial L}{\partial \ln(\bar{\kappa}_t^{nj,ij})} \mathbb{E}_t \hat{\kappa}_t^{nj,ij} + \sum_{i=1}^{N} \frac{\partial L}{\partial \ln(\bar{z}_t^{ij})} \mathbb{E}_t \hat{z}_t^{ij} \\
+ & \frac{1}{2} \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{w}_t^{ij}) \partial \ln(\bar{w}_t^{mj})} \mathbb{E}_t \hat{w}_t^{ij} \hat{w}_t^{mj} + \frac{1}{2} \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{P}_t^{ij}) \partial \ln(\bar{P}_t^{mj})} \mathbb{E}_t \hat{P}_t^{ij} \hat{P}_t^{mj} \\
+ & \frac{1}{2} \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{\kappa}_t^{nj,ij}) \partial \ln(\bar{\kappa}_t^{nj,mj})} \mathbb{E}_t \hat{\kappa}_t^{nj,ij} \hat{\kappa}_t^{nj,mj} + \frac{1}{2} \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{z}_t^{ij}) \partial \ln(\bar{z}_t^{mj})} \mathbb{E}_t \hat{z}_t^{ij} \hat{z}_t^{mj} \\
+ & \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{w}_t^{ij}) \partial \ln(\bar{P}_t^{mj})} \mathbb{E}_t \hat{w}_t^{ij} \hat{P}_t^{mj} + \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{P}_t^{ij}) \partial \ln(\bar{z}_t^{mj})} \mathbb{E}_t \hat{P}_t^{ij} \hat{z}_t^{mj} + \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{P}_t^{ij}) \partial \ln(\bar{\kappa}_t^{nj,mj})} \mathbb{E}_t \hat{P}_t^{ij} \hat{\kappa}_t^{nj,mj} \\
+ & \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{w}_t^{ij}) \partial \ln(\bar{\kappa}_t^{nj,mj})} \mathbb{E}_t \hat{w}_t^{ij} \hat{\kappa}_t^{nj,mj} + \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{w}_t^{ij}) \partial \ln(\bar{z}_t^{mj})} \mathbb{E}_t \hat{w}_t^{ij} \hat{z}_t^{mj} + \sum_{i,m}^{N} \frac{\partial^2 L}{\partial \ln(\bar{\kappa}_t^{nj,ij}) \partial \ln(\bar{z}_t^{mj})} \mathbb{E}_t \hat{\kappa}_t^{nj,ij} \hat{z}_t^{mj},
\end{aligned}
\tag{D.43}
$$

where

$$
\begin{aligned}
\frac{\partial L}{\partial \ln(\bar{w}_t^{kj})} &= \gamma^{kj}\bar{\lambda}_t^{nj,kj}, \quad \frac{\partial L}{\partial \ln(\bar{P}_t^{kj})} = (1-\gamma^{kj})\bar{\lambda}_t^{nj,kj}, \\
\frac{\partial L}{\partial \ln(\bar{\kappa}_t^{nj,ij})} &= \bar{\lambda}_t^{nj,ij}, \quad \frac{\partial L}{\partial \ln(\bar{z}_t^{kj})} = -\frac{1}{\theta^j}\bar{\lambda}_t^{nj,kj}, \\
\frac{\partial^2 L}{\partial \ln(\bar{w}_t^{kj})\partial \ln(\bar{w}_t^{mj})} &= \mathbb{1}(m=k)(\gamma^{kj})^2(-\theta^j)\bar{\lambda}_t^{nj,kj} - \gamma^{kj}\gamma^{mj}(-\theta^j)\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= (-\theta^j)\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k)(\gamma^{kj})^2 - \gamma^{kj}\gamma^{mj}\bar{\lambda}_t^{nj,kj}\Big), \\
\frac{\partial^2 L}{\partial \ln(\bar{P}_t^{kj})\partial \ln(\bar{P}_t^{mj})} &= \mathbb{1}(m=k)(1-\gamma^{kj})^2(-\theta^j)\bar{\lambda}_t^{nj,mj} - (1-\gamma^{kj})(1-\gamma^{mj})(-\theta^j)\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= (-\theta^j)\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k)(1-\gamma^{kj})^2 - (1-\gamma^{kj})(1-\gamma^{mj})\bar{\lambda}_t^{nj,kj}\Big), \\
\frac{\partial^2 L}{\partial \ln(\bar{\kappa}_t^{nj,kj})\partial \ln(\bar{\kappa}_t^{nj,mj})} &= \mathbb{1}(m=k)(-\theta^j)\bar{\lambda}_t^{nj,mj} - (-\theta^j)\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= (-\theta^j)\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big) \\
\frac{\partial^2 L}{\partial \ln(\bar{z}_t^{kj})\partial \ln(\bar{z}_t^{mj})} &= -\frac{1}{\theta^j}\mathbb{1}(m=k)\bar{\lambda}_t^{nj,mj} - \Big(-\frac{1}{\theta^j}\Big)\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= -\frac{1}{\theta^j}\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big), \\
\frac{\partial^2 L}{\partial \ln(\bar{w}_t^{kj})\partial \ln(\bar{P}_t^{mj})} &= \mathbb{1}(m=k)\gamma^{kj}(1-\gamma^{mj})(-\theta^j)\bar{\lambda}_t^{nj,mj} - \gamma^{kj}(1-\gamma^{mj})(-\theta^j)\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= (-\theta^j)\gamma^{kj}(1-\gamma^{mj})\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big), \\
\frac{\partial^2 L}{\partial \ln(\bar{P}_t^{kj})\partial \ln(\bar{z}_t^{mj})} &= (1-\gamma^{kj})\mathbb{1}(m=k)\bar{\lambda}_t^{nj,mj} - (1-\gamma^{kj})\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= (1-\gamma^{kj})\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big), \\
\frac{\partial^2 L}{\partial \ln(\bar{P}_t^{kj})\partial \ln(\bar{\kappa}_t^{nj,mj})} &= (1-\gamma^{kj})(-\theta^j)\mathbb{1}(m=k)\bar{\lambda}_t^{nj,mj} - (1-\gamma^{kj})(-\theta^j)\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= (-\theta^j)(1-\gamma^{kj})\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big), \\
\frac{\partial^2 L}{\partial \ln(\bar{w}_t^{kj})\partial \ln(\bar{\kappa}_t^{nj,mj})} &= \gamma^{kj}(-\theta^j)\mathbb{1}(m=k)\bar{\lambda}_t^{nj,mj} - \gamma^{kj}(-\theta^j)\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= (-\theta^j)\gamma^{kj}\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - (-\theta^j)\bar{\lambda}_t^{nj,kj}\Big), \\
\frac{\partial^2 L}{\partial \ln(\bar{w}_t^{kj})\partial \ln(\bar{z}_t^{mj})} &= \gamma^{kj}\mathbb{1}(m=k)\bar{\lambda}_t^{nj,mj} - \gamma^{kj}\bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= \gamma^{kj}\bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big), \text{ and} \\
\frac{\partial^2 L}{\partial \ln(\bar{\kappa}_t^{nj,kj})\partial \ln(\bar{z}_t^{mj})} &= \mathbb{1}(m=k)\bar{\lambda}_t^{nj,mj} - \bar{\lambda}_t^{nj,kj}\bar{\lambda}_t^{nj,mj} \\
&= \bar{\lambda}_t^{nj,mj}\Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj}\Big).
\end{aligned}
$$

Second-order approximation to ideal price index

First order approximation for ideal price index is exact. Therefore, a second-order approximation of

$\mathbb{E}_t \hat{P}^n_t$ is:

$$\begin{aligned}\mathbb{E}_t \hat{P}^n_t &\approx \prod_{j=1}^{J} \Big(\frac{\partial L}{\partial \ln(\bar{P}^{nj}_t)} \mathbb{E}_t \hat{P}^{nj}_t \Big) \\ &= \sum_{j=1}^{J} \alpha^{nj} \mathbb{E}_t \hat{P}^{nj}_t. \end{aligned} \tag{D.44}$$

Second-order approximation to current account

Let $\ln\left(X^{ij}_t\right) = \ln\left((1-\gamma^{ij}) \sum_{n=1}^{N} \lambda^{nj,ij}_t X^{nj}_t + \alpha^{ij} \sum_k w^{ik}_t(z^t) l^{ik}_t\right) \equiv O(\boldsymbol{\lambda^{j,ij}_t}, \mathbf{X^j_t}, \mathbf{w^i_t}, \mathbf{l^i_t})$. Then, a second-order approximation of $\mathbb{E}_t \hat{X}^{ij}_t$ is:

$$\begin{aligned}\mathbb{E}_t \hat{X}^{ij}_t &\approx \sum_{n=1}^{N} \Big(\frac{\partial O}{\partial \ln(\bar{\lambda}^{nj,ij}_t)} \mathbb{E}_t \hat{\lambda}^{nj,ij}_t + \frac{\partial O}{\partial \ln(\bar{X}^{nj}_t)} \mathbb{E}_t \hat{X}^{nj}_t \Big) + \sum_{k=1}^{J} \Big(\frac{\partial L}{\partial \ln(\bar{w}^{ik}_t)} \mathbb{E}_t \hat{w}^{ik}_t + \frac{\partial O}{\partial \ln(\bar{l}^{ik}_t)} \mathbb{E}_t \hat{l}^{ik}_t \Big) \\
&+ \frac{1}{2} \sum_{n,m}^{N} \frac{\partial^2 O}{\partial \ln(\bar{\lambda}^{nj,ij}_t) \partial \ln(\bar{\lambda}^{mj,ij}_t)} \mathbb{E}_t \hat{\lambda}^{nj,ij}_t \hat{\lambda}^{mj,ij}_t + \frac{1}{2} \sum_{n,m}^{N} \frac{\partial^2 O}{\partial \ln(\bar{X}^{nj}_t) \partial \ln(\bar{X}^{mj}_t)} \mathbb{E}_t \hat{X}^{nj}_t \hat{X}^{mj}_t \\
&+ \sum_{k,s}^{J} \frac{1}{2} \frac{\partial^2 O}{\partial \ln(\bar{w}^{ik}_t) \partial \ln(\bar{w}^{is}_t)} \mathbb{E}_t \hat{w}^{ik}_t \hat{w}^{is}_t + \sum_{k,s}^{J} \frac{1}{2} \frac{\partial^2 O}{\partial \ln(\bar{l}^{ik}_t) \partial \ln(\bar{l}^{is}_t)} \mathbb{E}_t \hat{l}^{ik}_t \hat{l}^{is}_t \\
&+ \sum_{n,m}^{N} \frac{\partial^2 O}{\partial \ln(\bar{\lambda}^{nj,ij}_t) \partial \ln(\bar{X}^{mj}_t)} \mathbb{E}_t \hat{\lambda}^{nj,ij}_t \hat{X}^{mj}_t + \sum_{k,s}^{J} \frac{\partial^2 O}{\partial \ln(\bar{w}^{ik}_t) \partial \ln(\bar{l}^{is}_t)} \mathbb{E}_t \hat{w}^{ik}_t \hat{l}^{is}_t \\
&+ \sum_{n}^{N} \sum_{s}^{J} \frac{\partial^2 O}{\partial \ln(\bar{\lambda}^{nj,ij}_t) \partial \ln(\bar{w}^{is}_t)} \mathbb{E}_t \hat{\lambda}^{nj,ij}_t \hat{w}^{ij}_t + \sum_{n}^{N} \sum_{s}^{J} \frac{\partial^2 O}{\partial \ln(\bar{\lambda}^{nj,ij}_t) \partial \ln(\bar{l}^{is}_t)} \mathbb{E}_t \hat{\lambda}^{nj}_t \hat{l}^{is}_t \\
&+ \sum_{n}^{N} \sum_{s}^{J} \frac{\partial^2 O}{\partial \ln(\bar{X}^{nj}_t) \partial \ln(\bar{w}^{is}_t)} \mathbb{E}_t \hat{X}^{nj}_t \hat{w}^{ij}_t + \sum_{n}^{N} \sum_{s}^{J} \frac{\partial^2 O}{\partial \ln(\bar{X}^{nj}_t) \partial \ln(\bar{l}^{is}_t)} \mathbb{E}_t \hat{X}^{nj}_t \hat{l}^{is}_t, \end{aligned} \tag{D.45}$$

where

$$\begin{aligned}\frac{\partial O}{\partial \ln(\bar{\lambda}^{nj,ij}_t)} &= \frac{\partial O}{\partial \ln(\bar{X}^{nj}_t)} = (1-\gamma^{ij}) \bar{\varrho}^{nj,ij}_t, \\ \frac{\partial O}{\partial \ln(\bar{w}^{ik}_t)} &= \frac{\partial O}{\partial \ln(\bar{l}^{ik}_t)} = \alpha^{ij} \bar{\zeta}^{ik,ij}_t, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 O}{\partial \ln(\bar{X}_t^{nj})\partial \ln(\bar{X}_t^{mj})} &= (1-\gamma^{ij})\bar{\varrho}_t^{mj,ij}\Big(\mathbb{1}(m=n)-(1-\gamma^{ij})\bar{\varrho}_t^{nj,ij}\Big),\\
\frac{\partial^2 O}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{\lambda}_t^{mj,ij})} &= (1-\gamma^{ij})\bar{\varrho}_t^{mj,ij}\Big(\mathbb{1}(m=n)-(1-\gamma^{ij})\bar{\varrho}_t^{nj,ij}\Big),\\
\frac{\partial^2 O}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{X}_t^{mj})} &= (1-\gamma^{ij})\bar{\varrho}_t^{mj,ij}\Big(\mathbb{1}(m=n)-(1-\gamma^{ij})\bar{\varrho}_t^{nj,ij}\Big),\\
\frac{\partial^2 O}{\partial \ln(\bar{w}_t^{ik})\partial \ln(\bar{w}_t^{is})} &= \alpha^{ij}\bar{\zeta}_t^{ik,ij}\Big(\mathbb{1}(k=s)-\alpha^{ij}\bar{\zeta}_t^{is,ij}\Big),\\
\frac{\partial^2 O}{\partial \ln(\bar{l}_t^{ik})\partial \ln(\bar{l}_t^{is})} &= \alpha^{ij}\bar{\zeta}_t^{ik,ij}\Big(\mathbb{1}(k=s)-\alpha^{ij}\bar{\zeta}_t^{is,ij}\Big),\\
\frac{\partial O}{\partial \ln(\bar{w}_t^{ik})\partial \ln(\bar{l}_t^{is})} &= \alpha^{ij}\bar{\zeta}_t^{ik,ij}\Big(\mathbb{1}(k=s)-\alpha^{ij}\bar{\zeta}_t^{is,ij}\Big),\\
\frac{\partial^2 O}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{w}_t^{is})} &= -(1-\gamma^{ij})\alpha^{ij}\bar{\varrho}_t^{njij}\bar{\zeta}_t^{is,ij},\\
\frac{\partial^2 O}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{l}_t^{is})} &= -(1-\gamma^{ij})\alpha^{ij}\bar{\varrho}_t^{njij}\bar{\zeta}_t^{is,ij},\\
\frac{\partial^2 O}{\partial \ln(\bar{X}_t^{nj})\partial \ln(\bar{w}_t^{is})} &= -(1-\gamma^{ij})\alpha^{ij}\bar{\varrho}_t^{njij}\bar{\zeta}_t^{is,ij}, \text{ and}\\
\frac{\partial^2 O}{\partial \ln(\bar{X}_t^{nj})\partial \ln(\bar{l}_t^{is})} &= -(1-\gamma^{ij})\alpha^{ij}\bar{\varrho}_t^{njij}\bar{\zeta}_t^{is,ij}.
\end{aligned}
\tag{D.46}
$$

Second-order approximation to labor market clearing

Let $\ln\left(w_t^{ij}\right) = \ln\left(\gamma^{ij}\sum_{n=1}^{N}\frac{\lambda_t^{nj,ij}X_t^{nj}}{l_t^{ij}}\right) \equiv R(\lambda_t^{j,ij}, X_t^j, l_t^{ij})$. Then, a second-order approximation of $\mathbb{E}_t\hat{w}_{it}$ is:

$$
\begin{aligned}
\mathbb{E}_t\hat{w}_t^{ij} \approx\; & \sum_{n=1}^{N}\Big(\frac{\partial R}{\partial \ln(\bar{\lambda}_t^{nj,ij})}\mathbb{E}_t\hat{\lambda}_t^{nj,ij}+\frac{\partial R}{\partial \ln(\bar{X}_t^{nj})}\mathbb{E}_t\hat{X}_t^{nj}\Big)+\frac{\partial R}{\partial \ln(\bar{l}_t^{ij})}\mathbb{E}_t\hat{l}_t^{ij}\\
+\; & \frac{1}{2}\sum_{n,m}^{N}\frac{\partial^2 R}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{\lambda}_t^{mj,ij})}\mathbb{E}_t\hat{\lambda}_t^{nj,ij}\hat{\lambda}_t^{mj,ij}+\frac{1}{2}\sum_{n,m}^{N}\frac{\partial^2 R}{\partial \ln(\bar{X}_t^{nj})\partial \ln(\bar{X}_t^{mj})}\mathbb{E}_t\hat{X}_t^{nj}\hat{X}_t^{mj}\\
+\; & \frac{1}{2}\sum_{n,m}^{N}\frac{\partial^2 R}{\partial \ln(\bar{l}_t^{nj})\partial \ln(\bar{l}_t^{mj})}\mathbb{E}_t\hat{l}_t^{nj}\hat{l}_t^{mj}\\
+\; & \sum_{n,m}^{N}\frac{\partial^2 R}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{X}_t^{mj})}\mathbb{E}_t\hat{\lambda}_t^{nj,ij}\hat{X}_t^{mj}+\sum_{n,m}^{N}\frac{\partial^2 R}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{l}_t^{mj})}\mathbb{E}_t\hat{\lambda}_t^{nj}\hat{l}_t^{mj}\\
+\; & \sum_{n,m}^{N}\frac{\partial^2 R}{\partial \ln(\bar{X}_t^{nj})\partial \ln(\bar{l}_t^{mj})}\mathbb{E}_t\hat{X}_t^{nj}\hat{l}_t^{mj},
\end{aligned}
$$

where

$$\frac{\partial R}{\partial \ln(\bar{\lambda}_t^{nj,ij})} = \frac{\partial R}{\partial \ln(\bar{X}_t^{nj})} = \gamma^{ij}\bar{x}_t^{nj,ij},$$
$$\frac{\partial R}{\partial \ln(\bar{l}_t^{ij})} = -1,$$
$$\frac{\partial^2 R}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{\lambda}_t^{mj,ij})} = \gamma^{ij}\bar{x}_t^{nj,ij}\Big(\mathbb{1}(m=n) - \gamma^{ij}\bar{x}_t^{mj,ij}\Big),$$
$$\frac{\partial^2 R}{\partial \ln(\bar{X}_t^{nj})\partial \ln(\bar{X}_t^{mj})} = \gamma^{ij}\bar{x}_t^{nj,ij}\Big(\mathbb{1}(m=n) - \gamma^{ij}\bar{x}_t^{mj,ij}\Big),$$
$$\frac{\partial^2 R}{\partial \ln(l^{ij})\partial \ln(l^{ij})} = 0,$$
$$\frac{\partial^2 R}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{X}_t^{mj})} = \gamma^{ij}\bar{x}_t^{nj,ij}\Big(\mathbb{1}(m=n) - \gamma^{ij}\bar{x}_t^{mj,ij}\Big),$$
$$\frac{\partial^2 R}{\partial \ln(\bar{\lambda}_t^{nj,ij})\partial \ln(\bar{l}_t^{mj})} = 0, \text{ and}$$
$$\frac{\partial^2 R}{\partial \ln(\bar{X}_t^{nj})\partial \ln(\bar{l}_t^{mj})} = 0.$$

D.6.3 Closed-Form Expressions for Static Trade Equilibria, First Order

1. Invert the system of price equation (D.26) to obtain $\hat{P}_t^{nj}$ as a function of $\hat{w}_t^{ij}$ and changes in fundamentals:

$$\begin{aligned}\hat{P}_t^{nj} &= \sum_{i=1}^{N}\bar{\lambda}_t^{nj,ij}\left(\gamma^{ij}\hat{w}_t^{ij} + \left(1-\gamma^{ij}\right)\hat{P}_t^{ij} + \hat{\kappa}_t^{nj,ij} - \frac{1}{\theta^j}\hat{z}_t^{ij}\right)\\ \Rightarrow \hat{P}_t^{nj} &= \sum_{o=1}^{N}\Delta_t^{njoj}\Big[\sum_{i=1}^{N}\bar{\lambda}_t^{oj,ij}\left(\gamma^{ij}\hat{w}_t^{ij} + \hat{\kappa}_t^{oj,ij} - \frac{1}{\theta^j}\hat{z}_t^{ij}\right)\Big],\end{aligned} \tag{D.47}$$

where Δ_t denotes an $N \times J$ by $N \times J$ matrix of the Leontief-inverse of Θ_t, which is itself an $N \times J$ by $N \times J$ matrix. The $(n + (j-1) \times N)$-th row and $(o + (j-1) \times N)$-th column of Θ_t is defined as $\Theta_t^{njoj} \equiv \bar{\lambda}_t^{nj,oj}(1-\gamma^{oj})$. The $(n + (j-1) \times N)$-th row and $(o + (j-1) \times N)$-th column of Δ_t, which we denote by Δ_t^{njoj} is the $(n + (j-1) \times N)$-th row and $(o + (j-1) \times N)$-th column of $(\mathbf{I}_{N\times J} - \Theta_t)^{-1}$.

2. Rewrite $\hat{\lambda}_t^{nj,ij}$ defined in equation (D.25) as a function of $\hat{w}_t^{nj}$ and changes in fundamentals:

$$
\begin{aligned}
\hat{\lambda}_t^{nj,ij} &= -\theta^j\left(\gamma^{ij}\hat{w}_t^{ij} + \left(1-\gamma^{ij}\right)\hat{P}_t^{ij} - \hat{P}_t^{nj} + \hat{\kappa}_t^{nj,ij}\right) + \hat{z}_t^{ij} \\
&= -\theta^j\left(\gamma^{ij}\hat{w}_t^{ij} + \hat{\kappa}_t^{nj,ij}\right) + \hat{z}_t^{ij} - \theta^j\left(\left(1-\gamma^{ij}\right)\hat{P}_t^{ij} - \hat{P}_t^{nj}\right) \\
&\quad \text{plugging } \hat{P}_t^{ij} \text{ and } \hat{P}_t^{nj} \text{ from step 1} \\
\Rightarrow \hat{\lambda}_t^{nj,ij} &= -\theta^j\left(\gamma^{ij}\hat{w}_t^{ij} + \hat{\kappa}_t^{nj,ij}\right) + \hat{z}_t^{ij} \\
&\quad -\theta^j\left(1-\gamma^{ij}\right)\sum_{o=1}^{N}\Delta_t^{ijoj}\sum_m \bar{\lambda}_t^{oj,mj}\left(\gamma^{mj}\hat{w}_t^{mj} + \hat{\kappa}_t^{oj,mj} - \frac{1}{\theta^j}\hat{z}_t^{mj}\right) \\
&\quad +\theta^j\sum_{o=1}^{N}\Delta_t^{njoj}\sum_m \bar{\lambda}_t^{oj,mj}\left(\gamma^{mj}\hat{w}_t^{mj} + \hat{\kappa}_t^{oj,mj} - \frac{1}{\theta^j}\hat{z}_t^{mj}\right) \\
&= -\theta^j\left(\gamma^{ij}\hat{w}_t^{ij} + \hat{\kappa}_t^{nj,ij}\right) + \hat{z}_t^{ij} \\
&\quad -\theta^j\sum_{o=1}^{N}\left(\left(1-\gamma^{ij}\right)\Delta_t^{ijoj} - \Delta_t^{njoj}\right)\sum_m \bar{\lambda}_t^{oj,mj}\left(\gamma^{mj}\hat{w}_t^{mj} + \hat{\kappa}_t^{oj,mj} - \frac{1}{\theta^j}\hat{z}_t^{mj}\right) \\
&= C_t^{ij}\hat{w}^{ij} + \sum_m^N D_t^{njim}\hat{w}_t^{mj} + (\hat{z}_t^{ij} - \theta^j\hat{\kappa}_t^{nj,ij}) + \sum_m E_t^{njim}\hat{z}_t^{mj} - \sum_o\sum_m F^{njimo}\hat{\kappa}_t^{oj,mj}, \qquad \text{(D.48)}
\end{aligned}
$$

where C_t denotes an $N \times J$ by 1 vector such that the $(i + (j-1) \times N)$-th row of C_t is defined by $C_t^{ij} \equiv -\theta^j\gamma^{ij}$. D_t denotes an $N \times J$ by $N \times N$ matrix such that the (nj, im)-th entry—the $(n + (j-1) \times N)$-th row and $(i + (m-1) \times N)$-th column—of D_t is defined by $D^{njim} \equiv -\theta^j \sum_{o=1}^N \left(\left(1-\gamma^{ij}\right)\Delta_t^{ijoj} - \Delta_t^{njoj}\right)\bar{\lambda}_t^{oj,mj}\gamma^{mj}$. E_t denotes an $N \times J$ by $N \times N$ matrix such that the (nj, im)-th entry is defined by $E_t^{njim} \equiv \sum_{o=1}^N \left(\left(1-\gamma^{ij}\right)\Delta_t^{ijoj} - \Delta_t^{njoj}\right)\bar{\lambda}_t^{oj,mj}$, and F_t denotes an $N \times J$ by $N \times N \times N$ matrix such that its (nj, imo)-th entry—the $(n + (j-1) \times N)$-th row and $(i + (m-1) \times N + (o-1) \times N \times N)$-th column of F_t—is defined by $F_t^{njimo} \equiv \theta^j\left(\left(1-\gamma^{ij}\right)\Delta_t^{ijoj} - \Delta_t^{njoj}\right)\bar{\lambda}_t^{oj,mj}$.

3. Solve for $\hat{X}_t^{ij}$ defined in equation (D.28) as a function of $\hat{w}_t^{nj}$, $\hat{l}_t^{ok}$, and changes in fundamentals

$$
\begin{aligned}
\hat{X}_t^{ij} &= \left(1-\gamma^{ij}\right)\sum_n^N \bar{\varrho}_t^{nj,ij}\left(\hat{\lambda}_t^{nj,ij} + \hat{X}_t^{nj}\right) + \alpha^{ij}\sum_k^J \bar{\zeta}_t^{ik,ij}\left(\hat{w}_t^{ik} + \hat{l}_t^{ik}\right) \\
\Rightarrow \hat{X}_t^{ij} &= \sum_o^N \Gamma_t^{ijoj}\Big[\sum_l^N\left(1-\gamma^{oj}\right)\bar{\varrho}_t^{lj,oj}\hat{\lambda}_t^{lj,oj} + \alpha^{oj}\sum_k^J \bar{\zeta}_t^{ok,oj}\left(\hat{w}_t^{ok} + \hat{l}_t^{ok}\right)\Big] \\
&= \sum_o^N\sum_l^N G_t^{ijol}\hat{\lambda}_t^{lj,oj} + \sum_o^N\sum_k^J H_t^{ijok}\left(\hat{w}_t^{ok} + \hat{l}_t^{ok}\right) \\
&\quad \text{plugging } \hat{\lambda}_t^{lj,oj} \text{ from step 2} \\
\Rightarrow \hat{X}_t^{ij} &= \sum_o^N\sum_l^N G_t^{ijol}\Big(C_t^{oj}\hat{w}^{oj} + \sum_m^N D_t^{ljom}\hat{w}_t^{mj} + (\hat{z}_t^{oj} - \theta^j\hat{\kappa}_t^{lj,oj}) + \sum_m E_t^{ljom}\hat{z}_t^{mj} - \sum_h^N\sum_m^N F_t^{ljomh}\hat{\kappa}_t^{hj,mj}\Big) \\
&\quad + \sum_o^N\sum_k^J H_t^{ijok}\left(\hat{w}_t^{ok} + \hat{l}_t^{ok}\right) \qquad \text{(D.49)}
\end{aligned}
$$

where Γ_t denotes an $N \times J$ by $N \times J$ matrix of the Leontief-inverse of U, such that (ij, oj)-th entry of Γ_t, denoted by Γ_t^{ijoj} is the (ij, oj)-th entry of $(\mathbf{I} - U_t)^{-1}$, and U_t denotes an $N \times J$ by $N \times J$ matrix such that (ij, oj)-th entry is defined by $U_t^{ijoj} \equiv (1-\gamma^{ij})\bar{\varrho}_t^{oj,ij}$. G_t denotes an $N \times J$ by $N \times N$ matrix such

that (ij, ol)-th entry is defined by $G_t^{ijol} \equiv \Gamma_t^{ijoj}\left(1-\gamma^{oj}\right)\bar{\varrho}_t^{lj,oj}$, and H_t denotes an $N \times J$ by $N \times J$ matrix such that (ij, ok)-th entry is defined by $H_t^{ijok} \equiv \Gamma_t^{ijoj}\alpha^{oj}\bar{\zeta}_t^{ok,oj}$.
4. Rewrite the labor market condition defined by equation (D.29) using the above expressions:

$$
\begin{aligned}
\hat{w}_t^{ij}+\hat{l}_t^{ij} &= \gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\left(\hat{\lambda}_t^{nj,ij}+\hat{X}_t^{nj}\right) \\
&= \gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\hat{\lambda}_t^{nj,ij}+\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\hat{X}_t^{nj} \\
&= \gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\Big(C_t^{ij}\hat{w}^{ij}+\sum_m^N D_t^{njim}\hat{w}_t^{mj}+(\hat{z}_t^{ij}-\theta^j\hat{\kappa}_t^{nj,ij})+\sum_m^N E_t^{njim}\hat{z}_t^{mj}-\sum_m^N\sum_h^N F_t^{njimh}\hat{\kappa}_t^{hj,mj}\Big) \\
&\quad +\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\Big(\sum_o^N\sum_l^N G^{njol}\Big(C_t^{oj}\hat{w}^{oj}+\sum_m^N D_t^{ljom}\hat{w}_t^{mj}+(\hat{z}_t^{oj}-\theta^j\hat{\kappa}_t^{lj,oj})+\sum_m^N E_t^{ljom}\hat{z}_t^{mj}-\sum_m^N\sum_h^N F_t^{ljomh}\hat{\kappa}_t^{hj,mj}\Big)\Big) \\
&\quad +\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\Big(\sum_o^N\sum_k^J H_t^{njok}\left(\hat{w}_t^{ok}+\hat{l}_t^{ok}\right)\Big) \\
&= \underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}C_t^{ij}}_{M_t^{*}}\hat{w}_t^{ij}+\underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\sum_m\Big(D_t^{njim}+\sum_o^N\sum_l^N G_t^{njol}D_t^{ljom}\Big)}_{M_t^{**}}\hat{w}_t^{mj} \\
&\quad +\underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\Big(\sum_o^N\sum_l^N G^{njol}C_t^{oj}}_{M_t^{***}}\hat{w}_t^{oj}\Big)+\underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}H_t^{njok}}_{\tilde{T}_t}\left(\hat{w}_t^{ok}+\hat{l}_t^{ok}\right) \\
&\quad +\underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}}_{Q_t^{*}}\hat{z}_t^{ij}+\underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\sum_m^N E_t^{njim}}_{Q_t^{**}}\hat{z}_t^{mj} \\
&\quad +\underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\sum_o^N\sum_l^N G_t^{njol}}_{Q_t^{***}}\hat{z}_t^{oj}+\underbrace{\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\sum_m^N\Big(\sum_o^N\sum_l^N G_t^{njol}E_t^{ljom}\Big)}_{Q_t^{****}}\hat{z}_t^{mj} \\
&\quad +\underbrace{-\theta^j\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}}_{F_t^{*}}\hat{\kappa}_t^{nj,ij}+\underbrace{-\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\sum_m^N\sum_h^N F_t^{njimh}}_{F_t^{**}}\hat{\kappa}_t^{hj,mj} \\
&\quad +\underbrace{-\theta^j\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\sum_o^N\sum_l^N G_t^{njol}}_{F_t^{***}}\hat{\kappa}_t^{lj,oj}+\underbrace{-\gamma^{ij}\sum_n^N \bar{\chi}_t^{nj,ij}\sum_m^N\sum_o^N\sum_l^N G_t^{njol}\sum_h^N F_t^{ljomh}}_{F_t^{****}}\hat{\kappa}_t^{hj,mj}
\end{aligned}
\tag{D.50}
$$

We stack the labor market clearing condition in matrix form and invert the equation to obtain $\hat{w}_t$:

$$
\begin{aligned}
\hat{w}_t+\hat{l}_t &= \tilde{M}_t\hat{w}_t+\tilde{T}_t(\hat{w}_t+\hat{l}_t)+\tilde{Q}_t\hat{z}_t+\tilde{F}_t\hat{K}_t \\
\Rightarrow \hat{w}_t &= [\mathbf{I}-\tilde{M}_t-\tilde{T}_t]^{-1}\Big((\tilde{T}_t-\mathbf{I})\hat{l}_t+\tilde{Q}_t\hat{z}_t+\tilde{F}_t\hat{K}_t\Big),
\end{aligned}
\tag{D.51}
$$

where $\hat{w}_t$, $\hat{l}_t$, and $\hat{z}_t$ are $(N\times J)$ by (1) vectors such that their respective $(ij,1)$-th entry is $\hat{w}_t^{ij}$, $\hat{l}_t^{ij}$, and $\hat{z}_t^{ij}$, and $\hat{K}_t$ is an $(N\times N\times J)$ by (1) vector such that $(nij,1)$-th entry of $\hat{K}_t$ – $(n+(i-1)\times N+(j-$

$1) \times N \times N)$-th row of vector $\hat{K}_t$ – is $\hat{\kappa}_t^{nj,ij}$. Other matrices are defined as

$$\begin{aligned}\tilde{M}_t &= M_t^* + M_t^{**} + M_t^{***}, \\ \tilde{Q}_t &= Q_t^* + Q_t^{**} + Q_t^{***} + Q_t^{****}, \text{ and} \\ \tilde{F}_t &= F_t^* + F_t^{**} + F_t^{***} + F_t^{****},\end{aligned} \tag{D.52}$$

where F_t^*, F_t^{**}, F_t^{***}, F_t^{****}, and $\tilde{F}_t$ are $(N \times J)$ by $(N \times N \times J)$ matrices defined below, in which the *(ijmhj)* superscript refers to the $(i + (j-1) \times N)$-th row and $(m + (h-1) \times N + (j-1) \times N \times N)$-th column of each matrices; and the dimension of other matrices is $(N \times J)$ by $(N \times J)$.

Each of these matrices is stacked according to equation (D.53):

$$\begin{aligned}
M_t^{*ijij} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} C_t^{ij}, \\
M_t^{**ijmj} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Big(D^{njim} + \sum_o^N \sum_l^N G_t^{njol} D_t^{ljom} \Big), \\
M_t^{***ijoj} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Big(\sum_l^N G^{njol} C_t^{oj} \Big), \\
\tilde{T}_t^{ijok} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} H_t^{njok} \\
Q_t^{*ijij} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij}, \\
Q_t^{**ijmj} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} E_t^{njim}, \\
Q_t^{***ijoj} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_l^N G_t^{njol}, \\
Q_t^{****ijmj} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Big(\sum_o^N \sum_l^N G_t^{njol} E_t^{ljom} \Big), \\
F_t^{*ijnij} &= -\theta^j \gamma^{ij} \bar{\chi}_t^{nj,ij}, \\
F_t^{**ijmhj} &= -\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} F_t^{njimh}, \\
F_t^{***ijolj} &= -\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} G_t^{njol}, \text{ and} \\
F_t^{****ijmhj} &= -\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \sum_l^N G_t^{njol} F_t^{ljomh}.
\end{aligned} \tag{D.53}$$

These matrices can be constructed using outcomes from any equilibrium. With these matrices at hand, we use equation (D.51) to solve the static trade equilibrium in each period for any guess of $\hat{l}$.

D.6.4 Closed-Form Expressions for Static Trade Equilibria, Second Order

The derivation of second-order closed-form expressions follows closely the one for first-order expressions, with the main difference being there are second-order terms appearing in each equation, which we denote using variables with a double tilde below. To make the comparison to first-order expression easier, we use red color for the second-order terms. The revised linear system of equations which takes into account these terms is:

$$
\begin{aligned}
\hat{P}_t^{nj} &= \sum_{i=1}^{N} \bar{\lambda}_t^{nj,ij} \left(\gamma^{ij} \hat{w}_t^{ij} + \left(1-\gamma^{ij}\right) \hat{P}_t^{ij} + \hat{\kappa}_t^{nj,ij} - \frac{1}{\theta^j} \hat{z}_t^{ij} \right) + \tilde{\tilde{P}}_t^{nj}, \\
\hat{\lambda}_t^{nj,ij} &= -\theta^j \left(\gamma^{ij} \hat{w}_t^{ij} + \left(1-\gamma^{ij}\right) \hat{P}_t^{ij} - \hat{P}_t^{nj} + \hat{\kappa}_t^{nj,ij} \right) + \hat{z}_t^{ij}, \\
\hat{X}_t^{ij} &= \left(1-\gamma^{ij}\right) \sum_{n}^{N} \bar{\varrho}_t^{nj,ij} \left(\hat{\lambda}_t^{nj,ij} + \hat{X}_t^{nj} \right) + \alpha^{ij} \sum_{k}^{J} \bar{\zeta}_t^{ik,ij} \left(\hat{w}_t^{ik} + \hat{l}_t^{ik} \right) + \tilde{\tilde{X}}_t^{ij}, \text{ and} \\
\hat{w}_t^{ij} + \hat{l}_t^{ij} &= \gamma^{ij} \sum_{n}^{N} \bar{\chi}_t^{nj,ij} \left(\hat{\lambda}_t^{nj,ij} + \hat{X}_t^{nj} \right) + \tilde{\tilde{w}}_t^{ij},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\tilde{P}}_t^{nj} &= \frac{1}{2} \sum_{k,m}^{N} (-\theta^j) \bar{\lambda}_t^{nj,mj} \gamma^{kj} \Big(\mathbb{1}(m=k) \gamma^{kj} - \gamma^{mj} \bar{\lambda}_t^{nj,kj} \Big) \hat{w}_t^{kj} \hat{w}_t^{mj} \\
&+ \frac{1}{2} \sum_{k,m}^{N} (-\theta^j) \bar{\lambda}_t^{nj,mj} (1-\gamma^{kj}) \Big(\mathbb{1}(m=k)(1-\gamma^{kj}) - (1-\gamma^{mj}) \bar{\lambda}_t^{nj,kj} \Big) \hat{P}_t^{kj} \hat{P}_t^{mj} \\
&+ \frac{1}{2} \sum_{k,m}^{N} \bar{\lambda}_t^{nj,mj} \Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj} \Big) \Big((-\theta^j) \hat{\kappa}_t^{nj,kj} \hat{\kappa}_t^{nj,mj} + \Big(-\frac{1}{\theta^j} \Big) \hat{z}_t^{kj} \hat{z}_t^{mj} + 2 \hat{\kappa}_t^{nj,kj} \hat{z}_t^{mj} \Big) \\
&+ \sum_{k,m}^{N} (-\theta^j) \bar{\lambda}_t^{nj,mj} \gamma^{kj} (1-\gamma^{mj}) \Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj} \Big) \hat{w}_t^{kj} \hat{P}_t^{mj} \\
&+ \sum_{k,m}^{N} \bar{\lambda}_t^{nj,mj} (1-\gamma^{kj}) \Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj} \Big) \Big(\hat{P}_t^{kj} \hat{z}_t^{mj} + (-\theta^j) \hat{P}_t^{kj} \hat{\kappa}_t^{nj,mj} \Big) \\
&+ \sum_{k,m}^{N} \bar{\lambda}_t^{nj,mj} \gamma^{kj} \Big(\mathbb{1}(m=k) - \bar{\lambda}_t^{nj,kj} \Big) \Big((-\theta^j) \hat{w}_t^{kj} \hat{\kappa}_t^{nj,mj} + \hat{w}_t^{kj} \hat{z}_t^{mj} \Big), \\
\tilde{\tilde{X}}_t^{ij} &= \frac{1}{2} \sum_{n,m}^{N} (1-\gamma^{ij}) \bar{\varrho}_t^{mjij} \Big(\mathbb{1}(m=n) - (1-\gamma^{ij}) \bar{\varrho}_t^{njij} \Big) \Big(\hat{\lambda}_t^{nj,ij} \hat{\lambda}_t^{mj,ij} + \hat{X}_t^{nj} \hat{X}_t^{mj} + 2 \hat{\lambda}_t^{nj,ij} \hat{X}_t^{mj} \Big) \\
&+ \frac{1}{2} \sum_{k,s}^{J} \alpha^{ij} \bar{\zeta}_t^{ik,ij} \Big(\mathbb{1}(k=s) - \alpha^{ij} \bar{\zeta}_t^{is,ij} \Big) \Big(\hat{w}_t^{ik} \hat{w}_t^{is} + \hat{l}_t^{ik} \hat{l}_t^{is} + 2 \hat{w}_t^{ik} \hat{l}_t^{is} \Big) \\
&+ \sum_{n}^{N} \sum_{s}^{J} \Big(-(1-\gamma^{ij}) \alpha^{ij} \bar{\varrho}_t^{njij} \bar{\zeta}_t^{is,ij} \Big) \Big(\hat{\lambda}_t^{nj,ij} \hat{w}_t^{ij} + \hat{\lambda}_t^{nj} \hat{l}_t^{is} + \hat{X}_t^{nj} \hat{w}_t^{ij} + \hat{X}_t^{nj} \hat{l}_t^{is} \Big), \text{ and} \\
\tilde{\tilde{w}}_t^{ij} &= \frac{1}{2} \sum_{n,m}^{N} \Big(\mathbf{1}(m=n) \gamma^{ij} \bar{\chi}_t^{nj,ij} - (\gamma^{ij})^2 \bar{\chi}_t^{mj,ij} \bar{\chi}_t^{nj,ij} \Big) \Big(\hat{\lambda}_t^{nj,ij} \hat{\lambda}_t^{mj,ij} + \hat{X}_t^{nj} \hat{X}_t^{mj} + 2 \hat{\lambda}_t^{nj,ij} \hat{X}_t^{mj} \Big).
\end{aligned}
$$

The variables with double tilde stand for the sum of *all* second-order terms appearing in the corresponding equations, rather than the variable corresponding to the letter. For example, $\tilde{\tilde{P}}_t^{nj}$ include not only the terms related to $\hat{P}_t^{kj} \hat{P}_t^{mj}$, but also other second-order terms appearing in the price equation.

1. Invert the system of price equation (D.26) to obtain $\hat{P}_t^{nj}$ as a second-order function of $\hat{w}_t^{ij}$ and

changes in fundamentals:

$$\begin{aligned}
\hat{P}_t^{nj} &= \sum_{i=1}^{N} \bar{\lambda}_t^{nj,ij} \left(\gamma^{ij} \hat{w}_t^{ij} + \left(1-\gamma^{ij}\right) \hat{P}_t^{ij} + \hat{\kappa}_t^{nj,ij} - \frac{1}{\theta^j} \hat{z}_t^{ij} \right) \\
\Rightarrow \hat{P}_t^{nj} &= \sum_{o=1}^{N} \Delta_t^{njoj} \Big[\sum_{i=1}^{N} \bar{\lambda}_t^{oj,ij} \left(\gamma^{ij} \hat{w}_t^{ij} + \hat{\kappa}_t^{oj,ij} - \frac{1}{\theta^j} \hat{z}_t^{ij} \right) + \tilde{\underline{P}}_t^{oj} \Big] \\
&= \sum_{o=1}^{N} \Delta_t^{njoj} \Big[\sum_{i=1}^{N} \bar{\lambda}_t^{oj,ij} \left(\gamma^{ij} \hat{w}_t^{ij} + \hat{\kappa}_t^{oj,ij} - \frac{1}{\theta^j} \hat{z}_t^{ij} \right) \Big] + \tilde{\tilde{\underline{P}}}_t^{nj},
\end{aligned} \tag{D.54}$$

where $\tilde{\tilde{\underline{P}}}_t$ is an $(N \times J)$ by (1) vector such that its (nj)-th entry—$(n + (j-1) \times N)$-th row—is defined by $\tilde{\tilde{\underline{P}}}_t^{nj} = \sum_{o=1}^{N} \Delta_t^{njoj} \tilde{\underline{P}}_t^{oj}$. Δ_t is defined in the paragraph below equation (D.47).

2. Rewrite $\hat{\lambda}_t^{nj,ij}$ as a second-order function of $\hat{w}_t^{nj}$ and changes in fundamentals:

$$\begin{aligned}
\hat{\lambda}_t^{nj,ij} &= -\theta^j \left(\gamma^{ij} \hat{w}_t^{ij} + \left(1-\gamma^{ij}\right) \hat{P}_t^{ij} - \hat{P}_t^{nj} + \hat{\kappa}_t^{nj,ij} \right) + \hat{z}_t^{ij} \\
&= -\theta^j \left(\gamma^{ij} \hat{w}_t^{ij} + \hat{\kappa}_t^{nj,ij} \right) + \hat{z}_t^{ij} - \theta^j \left(\left(1-\gamma^{ij}\right) \hat{P}_t^{ij} - \hat{P}_t^{nj} \right) \\
&\quad \text{plugging } \hat{P}_t^{ij} \text{ and } \hat{P}_t^{nj} \\
\Rightarrow \hat{\lambda}_t^{nj,ij} &= -\theta^j \left(\gamma^{ij} \hat{w}_t^{ij} + \hat{\kappa}_t^{nj,ij} \right) + \hat{z}_t^{ij} \\
&\quad -\theta^j \left(1-\gamma^{ij}\right) \sum_{o=1}^{N} \Delta_t^{ijoj} \sum_{m}^{N} \bar{\lambda}_t^{oj,mj} \left(\gamma^{mj} \hat{w}_t^{mj} + \hat{\kappa}_t^{oj,mj} - \frac{1}{\theta^j} \hat{z}_t^{mj} \right) \\
&\quad +\theta^j \sum_{o=1}^{N} \Delta_t^{njoj} \sum_{m}^{N} \bar{\lambda}_t^{oj,mj} \left(\gamma^{mj} \hat{w}_t^{mj} + \hat{\kappa}_t^{oj,mj} - \frac{1}{\theta^j} \hat{z}_t^{mj} \right) \\
&\quad -\theta^j \left(1-\gamma^{ij}\right) \tilde{\tilde{\underline{P}}}_t^{ij} + \theta^j \tilde{\tilde{\underline{P}}}_t^{nj} \\
&= -\theta^j \left(\gamma^{ij} \hat{w}_t^{ij} + \hat{\kappa}_t^{nj,ij} \right) + \hat{z}_t^{ij} \\
&\quad -\theta^j \sum_{o=1}^{N} \left(\left(1-\gamma^{ij}\right) \Delta_t^{ijoj} - \Delta_t^{njoj} \right) \sum_{m}^{N} \bar{\lambda}_t^{oj,mj} \left(\gamma^{mj} \hat{w}_t^{mj} + \hat{\kappa}_t^{oj,mj} - \frac{1}{\theta^j} \hat{z}_t^{mj} \right) \\
&\quad -\theta^j \left(1-\gamma^{ij}\right) \tilde{\tilde{\underline{P}}}_t^{ij} + \theta^j \tilde{\tilde{\underline{P}}}_t^{nj} \\
&= C_t^{ij} \hat{w}^{ij} + \sum_{m}^{N} D_t^{njim} \hat{w}_t^{mj} + (\hat{z}_t^{ij} - \theta^j \hat{\kappa}_t^{nj,ij}) + \sum_{m}^{N} E_t^{njim} \hat{z}_t^{mj} - \sum_{o}^{N} \sum_{m}^{N} F_t^{njimo} \hat{\kappa}_t^{oj,mj} \\
&\quad -\theta^j \left(1-\gamma^{ij}\right) \tilde{\tilde{\underline{P}}}_t^{ij} + \theta^j \tilde{\tilde{\underline{P}}}_t^{nj},
\end{aligned} \tag{D.55}$$

in which the matrices C_t, D_t, E_t, F_t are defined in the paragraph below equation (D.48).

3. Solve for $\hat{X}_t^{ij}$ as a second-order function of $\hat{w}_t^{nj}$, $\hat{l}_t^{ok}$, and changes in fundamentals:

$$\begin{aligned}
\hat{X}_t^{ij} &= \left(1-\gamma^{ij}\right)\sum_n^N \bar{\varrho}_t^{nj,ij}\left(\hat{\lambda}_t^{nj,ij}+\hat{X}_t^{nj}\right)+\alpha^{ij}\sum_k^J \bar{\zeta}_t^{ik,ij}\left(\hat{w}_t^{ik}+\hat{l}_t^{ik}\right)+\tilde{\tilde{X}}_t^{ij} \\
\Rightarrow \hat{X}_t^{ij} &= \sum_o^N \Gamma_t^{ijoj}\Big[\sum_l \left(1-\gamma^{oj}\right)\bar{\varrho}_t^{lj,oj}\hat{\lambda}_t^{lj,oj}+\alpha^{oj}\sum_k^J \bar{\zeta}_t^{ok,oj}\left(\hat{w}_t^{ok}+\hat{l}_t^{ok}\right)+\tilde{\tilde{X}}_t^{oj}\Big] \\
&= \sum_o^N\sum_l^N G_t^{ijol}\hat{\lambda}_t^{lj,oj}+\sum_o^N\sum_k^J H_t^{ijok}\left(\hat{w}_t^{ok}+\hat{l}_t^{ok}\right)+\sum_o^N \Gamma_t^{ijoj}\tilde{\tilde{X}}_t^{oj} \\
&= \sum_o^N\sum_l^N G_t^{ijol}\Big(C_t^{oj}\hat{w}^{oj}+\sum_m^N D_t^{ljom}\hat{w}_t^{mj}+(\hat{z}_t^{oj}-\theta^j\hat{\kappa}_t^{lj,oj}) \\
&\quad +\sum_m^N E_t^{ljom}\hat{z}_t^{mj}-\sum_h^N\sum_m^N F_t^{ljomh}\hat{\kappa}_t^{hj,mj}-\theta^j\left(1-\gamma^{oj}\right)\underline{\tilde{\tilde{P}}}_t^{oj}+\theta^j\underline{\tilde{\tilde{P}}}_t^{lj}\Big) \\
&\quad +\sum_o^N\sum_k^J H_t^{ijok}\left(\hat{w}_t^{ok}+\hat{l}_t^{ok}\right)+\underline{\tilde{\tilde{X}}}_t^{ij},
\end{aligned} \tag{D.56}$$

in which $\underline{\tilde{\tilde{X}}}_t$ is an $(N\times J)$ by (1) vector such that its (ij)-th entry is defined by $\underline{\tilde{\tilde{X}}}_t^{ij} \equiv \sum_o^N \Gamma^{ijoj}\tilde{\tilde{X}}_t^{oj}$. Matrices G_t and H_t are defined in the paragraph below equation (D.49).

4. Rewrite the labor market clearing condition using the above expressions:

$$
\begin{aligned}
\hat{w}_t^{ij} + \hat{l}_t^{ij} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \left(\hat{\lambda}_t^{nj,ij} + \hat{X}_t^{nj} \right) + \tilde{\bar{w}}_t^{ij} \\
&= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \hat{\lambda}_t^{nj,ij} + \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \hat{X}_t^{nj} + \tilde{\bar{w}}_t^{ij} \\
&= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Big(C_t^{ij} \hat{w}^{ij} + \sum_m^N D_t^{njim} \hat{w}_t^{mj} + (\hat{z}_t^{ij} - \theta^j \hat{\kappa}_t^{nj,ij}) + \sum_m^N E_t^{njim} \hat{z}_t^{mj} \\
&\quad - \sum_m^N \sum_h^N F_t^{njimh} \hat{\kappa}_t^{hj,mj} - \theta^j \left(1 - \gamma^{ij} \right) \underline{\tilde{P}}_t^{ij} + \theta^j \underline{\tilde{P}}_t^{nj} \Big) \\
&\quad + \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Big(\sum_o^N \sum_l^N G_t^{njol} \Big(C_t^{oj} \hat{w}^{oj} + \sum_m^N D_t^{ljom} \hat{w}_t^{mj} + (\hat{z}_t^{oj} - \theta^j \hat{\kappa}_t^{lj,oj}) \\
&\quad + \sum_m^N E_t^{ljom} \hat{z}_t^{mj} - \sum_m^N \sum_h^N F_t^{ljomh} \hat{\kappa}_t^{hj,mj} - \theta^j \left(1 - \gamma^{oj} \right) \underline{\tilde{P}}_t^{oj} + \theta^j \underline{\tilde{P}}_t^{lj} \Big) \Big) \\
&\quad + \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Big(\sum_o^N \sum_k^J H_t^{njok} \left(\hat{w}_t^{ok} + \hat{l}_t^{ok} \right) \Big) + \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \underline{\tilde{X}}_t^{nj} + \tilde{\bar{w}}_t^{ij} \\
&= \underbrace{\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} C^{ij}}_{M_t^{*}} \hat{w}_t^{ij} + \underbrace{\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_m^N \Big(D_t^{njim} + \sum_o^N \sum_l^N G_t^{njol} D_t^{ljom} \Big)}_{M_t^{**}} \hat{w}_t^{mj} \\
&\quad \underbrace{+ \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Big(\sum_o^N \sum_l^N G_t^{njol} C_t^{oj}}_{M_t^{***}} \hat{w}_t^{oj} \Big) + \underbrace{\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} H_t^{njok}}_{\tilde{T}_t} \left(\hat{w}_t^{ok} + \hat{l}_t^{ok} \right) \\
&\quad \underbrace{+ \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij}}_{Q_t^{*}} \hat{z}_t^{ij} + \underbrace{\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_m^N E_t^{njim}}_{Q_t^{**}} \hat{z}_t^{mj} \\
&\quad \underbrace{+ \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \sum_l^N G_t^{njol}}_{Q_t^{***}} \hat{z}_t^{oj} + \underbrace{\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_m^N \Big(\sum_o^N \sum_l^N G_t^{njol} E_t^{ljom} \Big)}_{Q_t^{****}} \hat{z}_t^{mj} \\
&\quad \underbrace{+ -\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij}}_{F_t^{*}} \hat{\kappa}_t^{nj,ij} + \underbrace{-\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_m \sum_h F_t^{njimh}}_{F_t^{**}} \hat{\kappa}_t^{hj,mj} \\
&\quad \underbrace{+ -\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \sum_l^N G_t^{njol}}_{F_t^{***}} \hat{\kappa}_t^{lj,oj} + \underbrace{-\gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_m^N \sum_o^N \sum_l^N G_t^{njol} \sum_h F_t^{ljomh}}_{F_t^{****}} \hat{\kappa}_t^{hj,mj} \\
&\quad \underbrace{+ -\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \left(1 - \gamma^{ij} \right) \sum_{o=1}^N \Delta_t^{ijoj}}_{R_t^{*}} \tilde{P}_t^{oj} + \underbrace{\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_{o=1}^N \Delta_t^{njoj}}_{R_t^{**}} \tilde{P}_t^{oj} \\
&\quad \underbrace{+ -\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \sum_l^N G_t^{njol} \left(1 - \gamma^{oj} \right) \sum_m^N \Delta_t^{ojmj}}_{R_t^{***}} \tilde{P}_t^{mj} + \underbrace{\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \sum_l^N G_t^{njol} \sum_m^N \Delta_t^{ljmj}}_{R_t^{****}} \tilde{P}_t^{mj} \\
&\quad \underbrace{+ \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \Gamma_t^{njoj}}_{\tilde{S}_t} \tilde{X}_t^{oj} + \tilde{\bar{w}}_t^{ij}.
\end{aligned}
\tag{D.57}
$$

We stack the labor market clearing condition in matrix form and invert the equation to obtain $\hat{w}_t$:

$$\begin{aligned} \hat{w}_t + \hat{l}_t &= \tilde{M}_t \hat{w}_t + \tilde{T}_t(\hat{w}_t + \hat{l}_t) + \tilde{Q}_t \hat{z}_t + \tilde{F}_t \hat{K}_t + \tilde{\tilde{w}}_t + \tilde{R}\tilde{\tilde{P}}_t + \tilde{S}_t \tilde{\tilde{X}}_t \\ \Rightarrow \hat{w}_t &= [\mathbf{I} - \tilde{M}_t - \tilde{T}_t]^{-1}\Big((\tilde{T}_t - \mathbf{I})\hat{l}_t + \tilde{Q}_t \hat{z}_t + \tilde{F}_t \hat{K}_t + \tilde{\tilde{w}}_t + \tilde{R}_t \tilde{\tilde{P}}_t + \tilde{S}_t \tilde{\tilde{X}}_t\Big), \end{aligned} \tag{D.58}$$

where $\tilde{\tilde{w}}_t$, $\tilde{\tilde{P}}_t$, and $\tilde{\tilde{X}}_t$ are $(N \times J)$ by (1) vectors such that their $(ij, 1)$-th entry is $\tilde{\tilde{w}}_t^{ij}$, $\tilde{\tilde{P}}_t^{ij}$, and $\tilde{\tilde{X}}_t^{ij}$, respectively.

To solve the static trade equilibrium with second-order accuracy, we use the results in Section D.6.3 to obtain a first-order solution, with which we construct the second-order terms, represented by red double tilde variables in equation (D.58). In particular, $\tilde{T}_t$, $\tilde{M}_t$, $\tilde{Q}_t$, and $\tilde{F}_t$ are identical to the first order case (equation (D.52)), while $\tilde{R}_t$ and $\tilde{S}_t$ contain the second-order effects, where the dimension of $\tilde{R}_t$ and $\tilde{S}_t$ is $(N \times J)$ by $(N \times J)$. Each of these matrices is stacked according to equation (D.59):

$$\begin{aligned} \tilde{R}_t &= R_t^{*} + R_t^{**} + R_t^{***} + R_t^{****}, \text{ where} \\ R_t^{*ijoj} &= -\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \left(1 - \gamma^{ij}\right) \Delta_t^{ijoj}, \\ R_t^{**ijoj} &= \theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Delta_t^{njoj}, \\ R_t^{***ijmj} &= -\theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \sum_l^N G_t^{njol} \left(1 - \gamma^{oj}\right) \Delta_t^{ojmj}, \\ R_t^{****ijmj} &= \theta^j \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \sum_o^N \sum_l^N G_t^{njol} \Delta_t^{ljmj}, \text{ and} \\ \tilde{S}_t^{ijoj} &= \gamma^{ij} \sum_n^N \bar{\chi}_t^{nj,ij} \Gamma_t^{njoj}. \end{aligned} \tag{D.59}$$

Again, these matrices can be constructed using outcomes from the deterministic allocation. We then invert the rest of the equation treating the second-order terms ($\tilde{\tilde{w}}_t$, $\tilde{\tilde{P}}_t$, and $\tilde{\tilde{X}}_t$) as constants. The solution has second-order accuracy.